Algebraic $K$-Theory and Functional Analysis

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Around 1978 Max Karoubi made a striking conjecture about the $K$-theory of Banach algebras. His conjecture predicted that the topological $K$-groups $K^\text{top}_*(B)$ of a unital Banach algebra $B$ were isomorphic to the algebraic $K$-groups $K_*(B\hat{\otimes}_\pi \mathcal{K})$ of the Grothendieck’s projective tensor product of $B$ and the ideal $\mathcal{K}$ of compact operators on the standard Hilbert space $H = \ell^2(\mathbb{N})$. Notice that $B\hat{\otimes}_\pi \mathcal{K}$ is a norm completion of the matrix algebra $M_\infty(B) = B \otimes M_\infty(\mathbb{C})$ and $K_*(M_\infty(B)) = K_*(B)$.

A few years later, Alain Connes developed his Noncommutative Differential Geometry [3] which puts $K$-theory in an even more prominent position. Because of the role played in Connes’ theory by Schatten and other normed ideals of operators on the Hilbert space, the need to determine the algebraic $K$-groups of such ideals is especially acute. An interest in these groups is further amplified by the fact that they can be viewed as canonical receptors of “higher index invariants.”

Karoubi’s Conjecture was finally proved in 1990 and the determination of the $K$-theory of Banach operator ideals has been recently reduced to a calculation of suitable cyclic homology groups [23]. I report on these and related developments below.

1. Bott periodicity

A fundamental property of the topological $K$-theory on the category of Banach algebras is its Bott periodicity [28]. A modern proof of the Bott periodicity might go as follows. For a unital Banach algebra $B$ the obvious pairing

$$B \times \mathbb{C} \longrightarrow B \otimes_{\mathbb{C}} \mathbb{C} = B$$

induces pairings between topological $K$-groups

$$K^\text{top}_i(B) \times K^\text{top}_j(\mathbb{C}) \longrightarrow K^\text{top}_{i+j}(B) \quad (i, j \in \mathbb{Z})$$

which transform $K^\text{top}_*(\mathbb{C}) = \bigoplus_{j \in \mathbb{Z}} K^\text{top}_j(\mathbb{C})$ into a graded ring and $K^\text{top}_*(B)$ into a graded right $K^\text{top}_*(\mathbb{C})$-module. Since $K^\text{top}_0(\mathbb{C}) = \mathbb{Z}$, $K^\text{top}_1(\mathbb{C}) = 0$, and since there exist such $u \in K^\text{top}_2(\mathbb{C})$ and $v \in K^\text{top}_{-2}(\mathbb{C})$ that $uv = vu = 1$, we must have $K^\text{top}_*(\mathbb{C}) \simeq \mathbb{Z}[u, u^{-1}]$. In particular, the multiplication by $v$
defines canonical isomorphisms

\[ K_i^{\text{top}}(B) \xrightarrow{\sim} K_{i-2}^{\text{top}}(B) \quad (i \in \mathbb{Z}). \]

How does this compare to the situation in algebraic \( K \)-theory? Pairing (1) again induces on \( K_\ast(C) = \bigoplus_{i \in \mathbb{Z}} K_i(C) \) a structure of a graded ring and on \( K_\ast(B) = \bigoplus_{i \in \mathbb{Z}} K_i(B) \), where \( B \) is an arbitrary unital \( \mathbb{C} \)-algebra, a structure of a graded \( K_\ast(C) \)-module. However, \( K_i(C) = 0 \) for all \( i < 0 \), so the only invertible elements in \( K_\ast(C) \) are \( \pm 1 \in K_0(C) = \mathbb{Z} \).

Let us go back to the topological \( K \)-theory and consider instead the following pairing

\[ B \times K \longrightarrow B \hat{\otimes}_\pi K. \tag{2} \]

where \( K = K(H) \) denotes the Banach algebra of compact operators on the Hilbert space \( H = \ell^2(\mathbb{N}) \). Since it is not difficult to show that the inclusion

\[ \bigcup_{n \geq 1} GL^{\text{top}}(B \otimes \mathbb{C} M_n(C)) \hookrightarrow GL^{\text{top}}(B \hat{\otimes}_\pi K) \]

is a weak homotopy equivalence, one can use (2) to define pairings

\[ K_i^{\text{top}}(B) \times K_j^{\text{top}}(K) \longrightarrow K_{i+j}^{\text{top}}(B) \quad (i, j \in \mathbb{Z}) \]

which transform \( K_\ast^{\text{top}}(K) = \bigoplus_{i \in \mathbb{Z}} K_i^{\text{top}}(K) \) into an associative graded ring and \( K_\ast^{\text{top}}(B) \) into a graded \( K_\ast^{\text{top}}(K) \)-module. Since \( K_0^{\text{top}}(C) = K_0^{\text{top}}(K) = \mathbb{Z} \), the image of \( u \in K_2^{\text{top}}(C) \) in \( K_2^{\text{top}}(K) \) and of \( v \in K_{-2}^{\text{top}}(C) \) in \( K_{-2}^{\text{top}}(K) \) must be mutual inverses. This implies that \( K_\ast^{\text{top}}(C) \xrightarrow{\sim} K_\ast^{\text{top}}(K) \simeq \mathbb{Z}[u, u^{-1}] \).

In sharp contrast with the case of the field of complex numbers, one has \( K_{-2}(K) \xrightarrow{\sim} K_{-2}^{\text{top}}(K) = \mathbb{Z} \) and \( K_2(K) \rightarrow K_2^{\text{top}}(K) \). If one could use the obvious inclusion

\[ K(C(\ell^2(\mathbb{N}))) \otimes_{\mathbb{C}} K(C(\ell^2(\mathbb{N}))) \hookrightarrow K(C(\ell^2(\mathbb{N} \times \mathbb{N}))) \]

to define pairings

\[ K_i(B \hat{\otimes}_\pi K) \times K_j(K) \longrightarrow K_{i+j}(B \hat{\otimes}_\pi K) \quad (i, j \in \mathbb{Z}) \]

and if those pairings made \( K_\ast(K) = \bigoplus_{j \in \mathbb{Z}} K_\ast(K) \) a graded associative ring and \( K_\ast(B \hat{\otimes}_\pi K) \) a graded \( K_\ast(K) \)-module then one would necessarily have the isomorphisms

\[ K_i(B \hat{\otimes}_\pi K) \xrightarrow{\sim} K_i^{\text{top}}(B \hat{\otimes}_\pi K) \simeq K_i^{\text{top}}(B) \quad (i \in \mathbb{Z}). \]
It is exactly this statement which has been conjectured by Max Karoubi in [13]. In the introductory part of this Section I attempted to reconstruct the line of argument that might have led Karoubi to the formulation of his conjecture.

The case $i \leq 0$ has been proved by Karoubi in [13]. On the other hand, the case $i \geq 3$ of Karoubi’s Conjecture had not been verified for a single Banach algebra until 1990 when Karoubi’s Conjecture was finally proved [23] (in an even greater generality than anticipated by him) following the resolution of the excision problem in rational algebraic $K$-theory [18], [19].

**Theorem 1.** Let $B$ be a Banach algebra with a left or right bounded approximate unit. Then the canonical comparison maps

$$K_i(B\hat{\otimes}_\pi\mathcal{K}) \longrightarrow K_i^{\text{top}}(B\hat{\otimes}_\pi\mathcal{K}) \quad (i \in \mathbb{Z})$$

are isomorphisms.

The above result admits a number of interesting generalizations and variants, some of which are mentioned in our next theorem (see also [27]).

**Theorem 2.** The $\mathbb{Z}$-graded abelian group $K_*(A) = \bigoplus_{i \in \mathbb{Z}} K_i(A)$ is a graded $K_*(\mathcal{K})$-module (and thus $K_i(A) \simeq K_{i-2}(A)$, $i \in \mathbb{Z}$) if $A$ is a $C^*$-algebra of any of the following types:

(a) $A = B\hat{\otimes}_\pi\mathcal{K}$ where $B$ is a locally multiplicatively convex Fréchet algebra with a uniformly bounded left or right approximate unit (cf. [24, Section 6]),

(b) $A = B\hat{\otimes}_{\min}\mathcal{K}$ where $B$ is an arbitrary $C^*$-algebra and $\hat{\otimes}_{\min}$ denotes the completed “minimal” tensor product of $C^*$-algebras.

(c) $A = B\otimes_C\mathcal{K}$ where $B$ is an arbitrary $H$-unital $C^*$-algebra (cf. [23]).

**Remarks.** (1) The class of Fréchet algebras $B$ mentioned in part (a) of Theorem 2 contains, among others, the algebra $C^r(X)$, $0 \leq r \leq \infty$, of $C^r$-functions on an arbitrary $C^*$-manifold $X$ and the algebra $O^{an}(V)$ of holomorphic functions on an arbitrary Stein space $V \subset \mathbb{C}^n$.

(2) Part (b) of Theorem 2 implies that the comparison map

$$K_*(B\hat{\otimes}_{\min}\mathcal{K}) \longrightarrow K_*^{\text{top}}(B\hat{\otimes}_{\min}\mathcal{K})$$

is an isomorphism for every $C^*$-algebra $B$. This is yet another statement that circulated during the 1980’s under the name of Karoubi’s conjecture. Its proof by a different method, which relied on a previous work of Kasparov, Cuntz and Higson (see [9]), is contained in [19]. For partial results, cf. [12], [9].
2. Operator ideals

Non-zero proper two-sided ideals \( J \) in the ring \( B = B(H) \) of bounded linear operators on the Hilbert space \( H = \ell^2(N) \) form a lattice whose minimal element is the ideal \( \mathcal{F} = H \otimes H^* \) of finite rank operators and the maximal element is the ideal \( \mathcal{K} = H \otimes \bigoplus_{\nu \in \nu} H^* \) of compact operators [2, Theorems 1.7 and 1.4, respectively].

Let \( A \) be an arbitrary unital \( C \)-algebra. The Morita invariance of algebraic \( K \)-theory implies that the inclusion \( \mathbb{C} \hookrightarrow \mathcal{F} \) given by

\[
1 \mapsto \begin{pmatrix} 1 & 0 & \cdots \\ 0 & \vdots & \ddots \\ \vdots \\ \end{pmatrix} \in B(H)
\]

induces an isomorphism \( K_*(A) \cong K_*(A \otimes \mathcal{F}) \).\(^1\) On the opposite side of the spectrum, the group \( K_*(A \otimes \mathcal{K}) \) is Bott-periodic, according to Part (c) of Theorem 2. Surprisingly, the ideal \( J = \mathcal{K} \) of compact operators is not the only operator ideal for which the groups \( K_*(A \otimes J) \) are Bott periodic.

Let us recall the definition of Schatten ideals. For a given \( p > 0 \), the \( p \)-th \emph{Schatten ideal} \( C_p \) consists of such compact operators \( T \in \mathcal{K} \) that

\[
\sum_{i=0}^{\infty} \lambda_i(TT^*)^{p/2} < \infty.
\]

Here \( \{\lambda_i(TT^*)\} \) is the sequence of eigenvalues of the operator \( TT^* \). It will also be convenient to consider the following related ideals

\[
C_{p^+} = \bigcap_{q>p} C_q \quad (0 \leq p < \infty)
\]

and

\[
C_{p^-} = \bigcup_{q<p} C_q \quad (0 < p < \infty)
\]

as well as the ideal \( C_\infty = \bigcup_{p>0} C_p \). For an ideal \( J \) and a positive integer \( n \), \( J^n \) will denote the ideal-theoretic \( n \)-th power of \( J \), i.e., the ideal additively generated by \( n \)-tuple products of elements of \( J \).

**Theorem 3.**

(a) The natural inclusion

\[
C_\infty(H) \otimes C_\infty(H) \subset C_\infty(H \otimes H^2),
\]

where \( H \otimes H^2 = \ell^2(N \times N) \) is the Hilbert tensor square of \( H \), induces a graded ring structure on \( K_*(C_\infty) \).

\(^1\) In the rest of this article, \( \otimes \) will denote \( \otimes_C \).
(b) The inclusion \( C_\infty \subset \mathcal{K} \) induces an isomorphism of rings

\[
K_*(C_\infty) \sim K_*(\mathcal{K}) \simeq \mathbb{Z}[u, u^{-1}] .
\]

(c) For every \( H \)-unital \( \mathbb{C} \)-algebra \( A \) and an ideal \( J = J^2 \subset B(H) \) such that, for some \( p > 0 \), \( C_p(H) \otimes J(H) \subset J(H^\otimes 2) \), the group \( K_*(A \otimes J) \) becomes a graded \( K_*(C_\infty) \)-module.

**Corollary.** If \( J = J^2 \) and, for some \( p > 0 \), \( C_p(H) \otimes J(H) \subset J(H^\otimes 2) \), then:

(a) the inclusion \( J \subset \mathcal{K} \) induces an isomorphism

\[
K_*(J) \sim K_*(\mathcal{K}) \simeq \mathbb{Z}[u, u^{-1}] ,
\]

(b) for every \( H \)-unital \( \mathbb{C} \)-algebra \( A \), the multiplication by \( u^{-1} \in K_{-2}(C_p) \) defines canonical isomorphisms

\[
K_i(A \otimes J) \sim K_{i-2}(A \otimes J) .
\]

The special meaning of the condition \( J = J^2 \) is explained by the following theorem.

**Theorem 4.** (cf. [25], [19]) For a given non-zero \( H \)-unital \( \mathbb{C} \)-algebra \( A \), the ring \( A \otimes J \) satisfies excision in algebraic \( K \)-theory if and only if \( J = J^2 \).

For the terminology and the discussion of basic aspects of excision we refer the reader to [19].

In particular, if \( J = J^2 \) one has

\[
K_*(A \otimes J) \sim K_*(A \otimes B, A \otimes J) \quad (B = B(H)) .
\]

If \( J \neq J^2 \) one has to choose between the absolute \( K \)-groups \( K_*(A \otimes J) \) and the relative ones \( K_*(A \otimes B, A \otimes J) \). The latter have the following two advantages:

(a) they are isomorphic to the groups \( K_{*+1}(A \otimes B/J) \) which serve as receptors of higher index invariants,

(b) they are also easier to deal with.

For every \( J \subset B(H) \), the correspondence

\[
A \mapsto K_*^J(A) := K_*(A \otimes B, A \otimes J)
\]
defines a functor from the category of $H$-unital $C^*$-algebras to the category of graded abelian groups. The family of functors $\{K^J_i; J \subseteq B(H)\}$ provides a "continuous" interpolation between the algebraic $K$-theory ($J = \mathcal{F}$) and the Bott periodic $K$-theories of Theorem 3. Our next objective will be to determine the structure of the functors $K^J_i$ for a large class of intermediate ideals.

First, notice that for every ideal $J \subseteq B(H)$ and a positive integer $n$, there exists a unique ideal $I$ such that $I^n = J$ (cf. [7]). We shall denote this ideal by $J^{1/n}$. Let $J_\infty = \bigcup_{n \geq 1} J^{1/n}$. For $i \leq 0$, all four arrows in the following square

\[
\begin{array}{ccc}
K_i(A \otimes J) & \longrightarrow & K_i(A \otimes J_\infty) \\
| & & | \\
K_i(A \otimes B, A \otimes J) & \longrightarrow & K_i(A \otimes B, A \otimes J_\infty)
\end{array}
\]

are isomorphisms. Assume that $C_p(H) \otimes J_\infty(H) \subseteq J_\infty(H^\hat{\circ}2)$ for some $p > 0$. Since $J_\infty = (J_\infty)^2$, Theorem 3 combined with diagram (4) implies that

\[
K^J_\infty(A) \simeq \begin{cases} 
K_0(A \otimes J) & i \text{ even} \\
K_{-1}(A \otimes J) & i \text{ odd}
\end{cases}
\]

**Theorem 5.** Assume that $C_p(H) \otimes J_\infty(H) \subseteq J_\infty(H^\hat{\circ}2)$ for some $p > 0$. Then, for every $H$-unital $C^*$-algebra $A$ there exists a functorial (in $A$ and $J$) long exact sequence of abelian groups

\[
\ldots K_{-1}(A \otimes J) \rightarrow HC_{2j-1}(A \otimes B, A \otimes J) \rightarrow K^J_{2j}(A) \rightarrow K_0(A \otimes J) \rightarrow HC_{2j-2}(A \otimes B, A \otimes J) \rightarrow K^J_{2j-1}(A) \rightarrow K_{-1}(A \otimes J) \rightarrow \ldots
\]

$(j \in \mathbb{Z})$.

Here $HC_i^Q(A \otimes B, A \otimes J)$, $i \in \mathbb{Z}$, denote the relative cyclic homology groups over $\mathbb{Q}$ (i.e., calculated in the category of $\mathbb{Q}$-algebras). In particular, these groups are uniquely divisible and they vanish by definition for $i < 0$. 

[Image of the diagram]
The hypothesis of Theorem 5 is satisfied if $J$ is a Banach ideal, i.e., if $J$ is complete with respect to some norm defined on $\mathcal{F} = H \otimes H^*$ and the composition of operators defines bilinear continuous mappings $\mathcal{B}(H) \times J \to \mathcal{B}(H)$ and $J \times \mathcal{B}(H) \to \mathcal{B}(H)$.

The special case $A = \mathbb{C}$ deserves a special attention in view of the following corollary of Theorem 5.

**Theorem 6.** Assume that $C_p(H) \otimes J_\infty(H) \subset J_\infty(H^\otimes 2)$ for some $p > 0$. Then, for every $j \in \mathbb{Z}$, one has the following functorial five-term exact sequences:

\[
0 \to HC_{2j-1}^Q(\mathcal{B}, J) \to K_{2j}(\mathcal{B}, J) \to \mathbb{Z} \to HC_{2j-2}^Q(\mathcal{B}, J) \to K_{2j-1}(\mathcal{B}, J) \to 0.
\]

This corollary is obtained by combining Theorem 5 with the following Proposition which does not seem to be well known.

**Proposition.** For every ideal $J \subset \mathcal{B}(H)$, one has $K_{-1}(J) = 0$ and the inclusion $\mathbb{C} \hookrightarrow J$ given by correspondence (3) induces an isomorphism $\mathbb{Z} \cong K_0(\mathbb{C}) \cong K_0(J)$.

Returning to Theorem 5, we observe that the cyclic homology group $HC_{\ast}^Q(A \otimes B, A \otimes J)$ measures the deviation of $K_{\ast}(A)$ from being Bott periodic. Calculation of $HC_{\ast}^Q(A \otimes B, A \otimes J)$ is facilitated by the existence of a spectral sequence which is described in our next theorem.

**Theorem 7.** For every operator ideal $J \subset \mathcal{B}(H)$ and every unital $\mathbb{C}$-algebra $A$, there exists a functorial (in $A$ and $J$) spectral sequence $E^r_{pq} \Rightarrow HC_{p+q}^Q(A \otimes B, A \otimes J)$ whose $E^1$-term is given by

\[
E^1_{pq} = \begin{cases} 
H_{q-p}^Q(A \otimes B; A \otimes J^{p+1})_{\mathbb{Z}/(p+1)\mathbb{Z}} & p \geq 0 \\
0 & p < 0 
\end{cases}
\]

A word of explanation: $H_{\ast}^Q(A \otimes B; A \otimes J^{p+1})$ denotes the Hochschild homology of the $\mathbb{Q}$-algebra $A \otimes B$ with coefficients in the $A \otimes B$-bimodule $A \otimes J^{p+1}$ (not to be confused with the relative Hochschild group $HH_{\ast}^Q(A \otimes B, A \otimes J^{p+1})$). If $A = A' \otimes_{\mathbb{Q}} \mathbb{C}$, i.e., if $A$ is obtained by extension of scalars from some $\mathbb{Q}$-algebra $A'$, then

\[
H_{\ast}^Q(A \otimes B; A \otimes J^{p+1}) \simeq HH_{\ast}^Q(A') \otimes_{\mathbb{Q}} H_{\ast}^Q(B; J^{p+1}).
\]
The subscript $\mathbb{Z}/(p+1)\mathbb{Z}$ in (6) refers to taking coinvariants of a certain natural action of the cyclic group $\mathbb{Z}/(p+1)\mathbb{Z}$ on $H^Q_*(A \otimes B; A \otimes J_{p+1})$. The spectral sequence of Theorem 7 is a close relative of a spectral sequence considered by Daniel Quillen in [16].

One can show that, if $J = J^2$, this spectral sequence naturally identifies with Connes' spectral sequence

$$E_1^{pq} \simeq \begin{cases} H^Q_{q-p}(A \otimes J) & p \geq 0 \\ 0 & p < 0 \end{cases}$$

which converges to $HC^Q_{p+q}(A \otimes J)$.

Theorem 7 brings out the importance of groups $H^Q_*(B; J)$ for all kinds of operator ideals $J$. The following theorem is therefore a very useful tool in the analysis of the groups $H^Q_*(B; J)$.

**Theorem 8.** Let $I$ and $J$ be two ideals contained in a third ideal $J' \subset \mathcal{B}(H)$. Assume that $I(H) \otimes J(H) \subset J'(H^\otimes 2)$ and that $I$ satisfies the condition

$$\mathcal{F} \subseteq [I, \mathcal{B}(H)]. \quad (7)$$

Then the map $H^Q_*(B; J) \rightarrow H^Q_*(B; J')$ induced by the inclusion $J \subset J'$ is zero.

We propose to call an ideal $I$ satisfying (7) large.

The following ideals are large: $C_p$, $p > 1$; $C_{p^+}$, $p \geq 1$; $C_{p^-}$, $p > 1$. Therefore, the spectral sequence of Theorem 7 gives us

**Corollary.** One has:

$$HC^Q_i(B, C_p) \simeq \begin{cases} 0 & i \leq 2p - 3, \ p \in \mathbb{Z}, \text{or } i \leq 2|p| - 1, \ p \notin \mathbb{Z} \\ C_{1/\left[ C_1, B \right]} & i = 2p - 2, \ p \in \mathbb{Z} \\ \mathbb{C} & i = 2|p|, \ p \notin \mathbb{Z} \end{cases}$$

and

$$HC^Q_i(B, C_{p^+}) \simeq \begin{cases} 0 & i \leq 2|p| - 1 \\ \mathbb{C} & i = 2|p| \end{cases}$$

and

$$HC^Q_i(B, C_{p^-}) \simeq \begin{cases} 0 & i \leq 2p - 3, \ p \in \mathbb{Z}, \text{or } i \leq 2|p| - 1, \ p \notin \mathbb{Z} \\ \mathbb{C} & i = 2p - 2, \ p \notin \mathbb{Z}, \text{or } i = 2|p|, \ p \notin \mathbb{Z} \end{cases} .$$

The group $C_{1/\left[ C_1, B \right]}$ decomposes into the direct sum $\mathbb{C} \oplus C^0_1/\left[ C_1, B \right]$ where $C^0_1$ denotes the space of trace class operators whose trace equals zero. Gary Weiss [22], [21] discovered that the codimension of the $\mathbb{C}$-vector
subspace \([C_1, B]\) of \(C_1\) is uncountable. Recently, Nigel Kalton [10] proved that the space \([C_1, B]\) consists of operators \(T \in \mathcal{K}\) such that

\[
\sum_{n=1}^{\infty} \frac{|\lambda_1 + \cdots + \lambda_n|}{n} < \infty
\]

where \(\{\lambda_n\} = \text{Spec } T\) is the spectrum of \(T\) arranged so that

\[|\lambda_1| \geq |\lambda_2| \geq \ldots\]

By combining the information about the cyclic homology of Schatten and related ideals with Theorems 6 and 7 we obtain the following description of the corresponding lower-dimensional \(K\)-groups:

\[
K_i(B, C_p) = \begin{cases} 
\mathbb{Z} & i \text{ even} \\
0 & i \text{ odd}
\end{cases}
\]

for \(i \leq 2p - 2\) if \(p \in \mathbb{Z}\) and \(i \leq 2[p] \) if \(p \notin \mathbb{Z}\),

\[
K_i(B, C_{p+}) = \begin{cases} 
\mathbb{Z} & i \text{ even} \\
0 & i \text{ odd}
\end{cases}
\]

for \(i \leq 2[p]\) and

\[
K_i(B, C_{p-}) = \begin{cases} 
\mathbb{Z} & i \text{ even} \\
0 & i \text{ odd}
\end{cases}
\]

for \(i \leq 2p - 2\) if \(p \in \mathbb{Z}\) and \(i \leq 2[p]\) if \(p \notin \mathbb{Z}\). The isomorphisms of even-dimensional groups with \(\mathbb{Z}\) are induced by the inclusions of corresponding ideals into \(\mathcal{K}\). The higher dimensional \(K\)-groups are given by (5).

**Regulator maps.** Let \(J\) be any ideal satisfying the hypothesis of Theorem 6. Since the comparison maps \(K_{2j}(\mathbb{C}) \to K_{2j}^{\text{top}}(\mathbb{C}) = \mathbb{Z}\) are zero, for \(j > 0\), the inclusion \(\mathcal{F} \hookrightarrow J\) induces homomorphisms

\[
\rho_{2j}^J : K_{2j}(\mathbb{C}) \to HC_{2j-1}^Q(B, J)
\]

and

\[
\rho_{2j+1}^J : K_{2j+1}(\mathbb{C}) \to HC_{2j}^Q(B, J)/\text{Image}(\mathbb{Z})
\]

which we suggest to call \(J\)-regulator maps. For \(J = C_{j+}\), the maps \(\rho_n^J\) are zero if \(n < 2j + 1\) and the first possibly nonzero maps are:

\[
\rho_{2j+1} : K_{2j+1}(\mathbb{C}) \to HC_{2j}^Q(B, C_{j+})/\mathbb{Z} \simeq \mathbb{C}^* 
\] (8)
and

$$\rho_{2j+2} : K_{2j+2}(\mathbb{C}) \longrightarrow HC^{Q}_{2j+1}(\mathcal{B}, C_{j+}) \simeq H_{1}^{Q}(\mathcal{B}; C_{1+})/d^{1}(\mathbb{C})$$ (9)

where $d^{1}$ denotes the differential $d_{pp}^{1} : E_{p}^{1} \rightarrow E_{p-1,p}^{1}$ of the spectral sequence of Theorem 7. It is natural to expect that (8) coincides up to a normalization with the regulator maps defined by Beilinson [1]. On the other hand, the “secondary” regulators (9) seem to be entirely new.

Another potential class of “regulator maps” arises in connection with pseudo-differential operators. Let $CL_{m}(X), m \in \mathbb{C}$, denote the space of scalar pseudodifferential operators of classical type (see e.g., [17]) on a closed $C^{\infty}$-manifold $X$. The composition of operators makes $CL^{0}(X)$ into a subring of $\mathcal{B}(L^{2}(X))$. The principal symbol map $\sigma : CL^{0}(X) \rightarrow C^{\infty}(S^{*}X)$ identifies $C^{\infty}(S^{*}X)$ with the quotient ring $CL^{0}(X)/CL^{-1}(X)$. The ideal $CL^{-1}(X)$ is a principal ideal generated by $(1 + \Delta_{X})^{-1/2}$ where $\Delta_{X}$ is the Laplace operator associated with any Riemannian connection on $X$. Therefore the smallest ideal $J_{X} \subset \mathcal{B}(L^{2}(X))$ which contains $CL^{-1}(X)$ is also principal. We deduce from Hermann Weyl’s formula for the asymptotics of eigenvalues of $\Delta_{X}$ (see e.g., [17, Theorem 15.2]) that $J_{X}$ coincides with the ideal

$$S_{d} = \{ T \in \mathcal{K} | \lambda_{n}(TT^{*}) = O(n^{-2/d}) \}$$

where $d = \dim X$. In particular, the subrings $CL^{-1}(X) \subset \mathcal{B}(H)$, for any two $C^{\infty}$-manifolds of dimension $d$, span one and the same ideal in $\mathcal{B}(H)$.

The ideals $S_{d}$, $0 < d < \infty$, satisfy the hypothesis of Theorem 6 and, therefore, the $K$-groups $K_{*}(\mathcal{B}, S_{d})$ are given by (5). Since the natural map $H_{0}(\mathcal{B}; C_{1}) \rightarrow H_{0}(\mathcal{B}; S_{1})$ is zero, we get, in particular, that $K_{1}(\mathcal{B}, S_{1}) \simeq S_{1}/[S_{1}, \mathcal{B}]$. This last group is an infinite-dimensional complex vector-space. In fact, the set of $T \in \mathcal{K}$, for which

$$\|T\|_{\Omega} = \sup_{n} \frac{\sum_{i=0}^{n} \sqrt{\lambda_{i}(TT^{*})}}{\log n} < \infty,$$

where the eigenvalues of $TT^{*}$ are monotonically ordered $\lambda_{0} \geq \lambda_{1} \geq \ldots$, is a symmetrically normed Banach ideal known under the name of the dual Macaev ideal [15]. Let us denote it by $C_{\Omega}$. It contains $S_{1}$, though $S_{1}$ is not dense in $C_{\Omega}$, and neither is $\mathcal{F}$ dense in $S_{1}$. József Vârga [20] has

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3 This ideal owes its current prominence to Connes’ article [5]. (See also his forthcoming book [4].)
recently shown that the Banach space of \( \Omega \)-continuous traces on \( S_1 \) contains \((\ell^\infty/\ell^0)^*\).

The inclusion

\[
C^\infty(S^*X) = C\ell^0(X)/C\ell^{-1}(X) \hookrightarrow \mathcal{B}(L^2(X))/S_d(L^2(X))
\]

induces the corresponding homomorphisms on \( K \)-groups

\[
\iota_{jd} : K_j(C^\infty(S^*X)) \longrightarrow K_j(\mathcal{B}/S_d) \cong K_{j-1}(\mathcal{B}, S_d) \quad (d = \dim X)
\]

The map \( \iota_1d \) coincides with the index map \( K_1(C^\infty(S^*X)) \rightarrow \mathbb{Z} \). The odd \( K \)-groups \( K_{2i-1}(\mathcal{B}, S_d) \) are zero for \( i < d \) and the even \( K \)-groups \( K_{2d-1}(\mathcal{B}, S_d) \), \( i < d \), are isomorphic to \( \mathbb{Z} \). The first interesting group is \( K_{2d-1}(\mathcal{B}, S_d) \) which is isomorphic to \( S_1/[S_1, \mathcal{B}] \) modulo the image of \( Z = K_{2d}(\mathcal{K}) \). This image is zero for \( d = 1 \) and, in all likelihood, also for \( d > 1 \). It remains unclear whether the map

\[
\iota_{2d,d} : K_{2d}(C^\infty(S^*X)) \longrightarrow S_1/[S_1, \mathcal{B}] \quad (\text{mod image of } \mathbb{Z})
\]

is nonzero in general or not.

**References**


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