The present paper is devoted to the description of certain interactions between the algebraic $K$-theory and the theory of algebraic groups. It consists of two parts. In the first part we consider an application of the algebraic $K$-theory to the rationality problem for algebraic groups. In the second part, conversely, we show how the theory of algebraic groups allows to compute $K$-groups of algebraic varieties with relatively large automorphism group.

In the first section we recall basic definitions and results on rationality and stable rationality of algebraic group varieties. In the next section we define the group of equivalence classes, an important birational invariant of an algebraic group. This notion was introduced by Manin in [18] and studied for linear algebraic groups in [10] by Colliot-Thélène and Sansuc. In particular, the group of equivalence classes for algebraic tori was computed in this paper. We review this computation and also the stable birational classification of algebraic tori in section 3.

By a classical result of Weil [49], an adjoint simple group of classical series is the connected component of the automorphism group of some algebra with involution (except for some “non-classical” groups of the type $D_4$). In section 4 we give a computation of the group of equivalence classes for such algebraic groups. The result is given in terms of certain invariants of the corresponding algebra with involution.

In the following sections we consequently consider the classical types $A$, $B$, $C$ and $D$ of simply connected and adjoint absolutely simple groups. In some cases we compute the group of equivalence classes and prove the (stable) rationality of group varieties. For types $A$ and $D$ we give sufficient conditions for corresponding groups to be not stably rational. The algebraic $K$-theory comes out, for example, as the group $\text{SK}_1(A)$, being the group of equivalence classes of $\text{SL}_1(A)$, where $A$ is a central simple algebra. The proof of Theorem 5.4 is based on Rost’s computation of the group $\text{SK}_1$ for a biquaternion algebra, which, in its turn, uses in particular Swan’s computation of the higher algebraic $K$-theory of a projective quadric hypersurface. In the proof of Theorems 5.14 and 8.9 the general index reduction formula is used (see [26]), which is based on the computation of the
algebraic $K$-theory of certain projective homogeneous varieties given by Panin in [30].

By Corollary 1.8, algebraic groups of rank at most 2 are rational. In section 9, the complete stably rational classification of all absolutely simple groups of types $A_3 = D_3$ (rank 3) is given.

In the second part of the paper we consider the problem of the computation of the algebraic $K$-groups for algebraic varieties on which a “large” algebraic groups acts. The $K$-groups $K'_n(X)$ of the category of coherent sheaves on an algebraic variety $X$ are known at least for the following classes of varieties:

1. Homogeneous projective varieties [30].
2. Varieties of simply connected semisimple algebraic groups and principal homogeneous spaces over these groups [17], [29].
3. Toric varieties and toric models [27].

In all these cases there is a reductive algebraic group acting non-trivially on a variety. The basic objects of the second part of the present paper are reductive algebraic group $G$, defined over an arbitrary field $F$, and a $G$-scheme $X$, i.e. a scheme $X$ over $F$, on which $G$ acts. The problem we study here is to compare the equivariant $K$-groups $K'_n(G; X)$ of the category of $G$-modules on $X$ and the ordinary ones $K_n(X)$ of the category of coherent sheaves on $X$ via the restriction homomorphism

$$\text{res} : K'_n(G; X) \to K_n(X).$$

It turns out that sometimes the group $K'_n(G; X)$ is easy to compute. For example, if $X = G/H$ is a homogeneous variety with the natural action of $G$, where $H \subseteq G$ is a subgroup, then the category of $G$-modules on $X$ is equivalent to the category of finite dimensional representations of the group $H$. Since in the latter category any object has finite length, the computation of the $K$-groups of this category reduces to the computation of the $K$-groups of the endomorphism rings of simple objects. For example, $K'_0(G; G/H) = R(H)$ is the representation ring of $H$, which is a free abelian group generated by the classes of irreducible representations of $H$. Therefore, the knowledge of the kernel and cokernel of the restriction homomorphism sheds some light on the structure of $K'_n(X)$.

In the second part of the paper we review the main results of [24]. In section 10 we give certain basic definitions and results in the theory of algebraic groups. In particular, we give an algebraic definition of the fundamental group of an arbitrary algebraic group, which is not based on the choice of a maximal torus. In section 11 we introduce equivariant algebraic $K$-theory developed by Thomason in [42]. In the following section for a split reductive group $G$ with $\pi_1(G)$ torsion free and any $G$-scheme $X$ a spectral sequence with the $E^2$-term related to the equivariant $K$-groups of $X$, which converges to the ordinary $K$-groups of $X$, is given (Theorem 12.1). In particular, the restriction homomorphism induces an isomorphism

$$\mathbb{Z} \otimes_{R(G)} K'_0(G; X) \to K'_0(X).$$

For example,

$$K_0(G/H) \simeq \mathbb{Z} \otimes_{R(G)} R(H)$$

for any subgroup $H \subseteq G$. 
In section 13 we consider the case of reductive groups of inner type. We describe the kernel of the restriction homomorphism in terms of Tits algebras of the group $G$. At the end of the section we compute the group $K_0(G)$ for any adjoint semisimple group $G$ of inner type.

A simple argument (see the beginning of section 14) shows that the surjectivity of the restriction homomorphism for $n = 0$ and all $G$-schemes $X$ implies that the Picard group $\text{Pic}(G \otimes_F E)$ is trivial for any finite field extension $E/F$ (we call groups $G$ with this property factorial). It turns out that the converse statement holds for quasi-projective varieties (Theorem 14.1): If $G$ is factorial and $X$ is quasi-projective, then the restriction homomorphism is surjective for $n = 0$. In particular, for any subgroup $H \subseteq G$, the group $K_0(G/H)$ is generated by the classes of vector bundles $(G \times V)/H$ over $G/H$ for all linear representations $H \to \text{GL}(V)$ (Corollary 14.2). Note that for a projective homogeneous variety $X$ under a simply connected semisimple group $G$ the surjectivity of of the restriction homomorphism has been proven by I. Panin in [31, Th. 2].

The last section is devoted to some applications.

We use the following notation. For a central simple algebra $A$ over $F$ by $\text{Nrd} : A^\times \to F^\times$ we denote the reduced norm homomorphism. The degree $\deg(A)$ is the square root of the dimension of $A$ and index $\text{ind}(A)$ is the square root of the dimension of the division algebra similar to $A$. For a field extension $E/F$ the algebra $A \otimes_F E$ over $E$ is denoted by $A_E$.

$\text{Br}(F)$ is the Brauer group of a field $F$.

We use the word scheme for a separated scheme of finite type over a field. If $X$ is a scheme over $F$ and $E/F$ is a field extension, then the scheme $X \otimes_F E$ we simply denote by $X_E$ and the set of $E$-points $\text{Mor}_{\text{Spec} F}(\text{Spec} E, X)$ by $X(E)$. For a morphism of schemes $\alpha : X \to Y$ over $F$, the map of sets of points $X(E) \to Y(E)$ is denoted by $\alpha_E$.

If $E/F$ is a finite separable field extension and $Y$ is an algebraic variety defined over a field $E$, then $R_{E/F}(Y)$ is the variety over $F$, obtained from $Y$ by the restriction of scalars.

$\mathbb{A}_F^n$ is the affine space over $F$.

An algebraic group is a smooth affine group scheme of finite type over a field. The one-dimensional split torus $\text{Spec} F[t, t^{-1}]$ is denoted by $G_{m,F}$ and the additive group $\text{Spec} F[t]$ by $G_{a,F}$.

For an algebraic group $G$ over $F$ by $H^1(F, G)$ we denote the pointed set $H^1\left(\text{Gal}(F_{\text{sep}}/F), G(F_{\text{sep}})\right)$ ([39]).

1. Rational and stably rational algebraic groups

Let $F$ be an arbitrary field. An irreducible algebraic variety $X$, defined over $F$, is called rational if $X$ is birationally isomorphic to an affine space, or equivalently, the function field $F(X)$ is a purely transcendental extension of $F$. An irreducible algebraic variety $X$, defined over $F$, is called stably rational if the variety $X \times_F \mathbb{A}_F^n$ is rational for some $n$. 
If $G$ is a connected algebraic group, defined over a field $F$, we simply say that $G$ is (stably) rational if the variety of $G$ is (stably) rational. It is still unknown if there exists a stably rational group which is not rational.

The following well-known statement is often useful.

**Lemma 1.1.** Let $1 \rightarrow H \rightarrow G \rightarrow T \rightarrow 1$ be an exact sequence of algebraic groups defined over $F$. Assume that for any field extension $E/F$ the map $H^1(E, H) \rightarrow H^1(E, G)$ has trivial kernel. Then the sequence splits rationally, i.e. there exists a rational morphism $\psi : T \rightarrow G$, such that $\varphi \circ \psi = \text{id}_T$. In particular, $G$ is birationally isomorphic to the product $H \times T$.

**Proof.** Let $E = F(T)$ be the function field of $T$. It follows from the exactness of the sequence

$$G(E) \rightarrow T(E) \rightarrow H^1(E, H) \rightarrow H^1(E, G)$$

that the generic point in $T(E)$ belongs to the image of $G(E) \rightarrow T(E)$, hence the sequence splits rationally. □

A connected solvable group $G$ over $F$ is called *split* if there is a series of subgroups $G = G_0 \supset G_1 \supset \cdots \supset G_n = 1$ such that $G_i/G_{i+1}$ is isomorphic either to $G_a,F$ or to $G_m,F$ for $i = 0, 1, \ldots, n - 1$ (see [4, Def. 15.1]). Since $H^1(F, G_{a,F}) = 1 = H^1(F, G_{m,F})$, Lemma 1.1 gives

**Corollary 1.2.** ([4, Cor. 15.8]) A connected solvable split group is rational. □

**Example 1.3** A connected unipotent group over a perfect field splits ([4, Cor. 15.5(ii)]) and hence is rational.

An algebraic torus $T$ over $F$ is called *quasi-trivial* if it is isomorphic to $\text{GL}_1(C)$ for some étale $F$-algebra $C$. Since $T$ is an open subset in the affine space of $C$, it is a rational variety.

A reductive group $G$ over $F$ is called *split* (resp. *quasi-split*) if it has a maximal torus which splits over $F$ (resp. a Borel subgroup defined over $F$) (see [4, Th. 18.7]).

**Proposition 1.4.** ([4]) A split reductive group is rational.

**Proof.** Let $B$ be a Borel subgroup in $G$ defined over $F$, $B^-$ be the opposite Borel subgroup, $U$ (resp. $U^-$) the unipotent radical of $B$ (resp. of $B^-$), $T$ a maximal torus in $B$ defined over $F$. Then the product morphism

$$U \times T \times U^- \rightarrow G$$

is a birational isomorphism and $U, U^-$ and $T$ are split solvable groups. □

**Remark 1.5** The statement of the Proposition 1.4 is still valid for a quasi-split semisimple simply connected or adjoint group, since in this case $T$ is a quasi-trivial torus.

If $F$ is a perfect field, then the unipotent radical $U$ of $G$ is defined over $F$. Since $U$ splits, by induction $H^1(E, U) = 1$ for any field extension $E/F$ and, by
Lemma 1.1, the sequence $1 \to U \to G \to G/U \to 1$ splits rationally, hence $G$ is birationally isomorphic to the product $U \times (G/U)$, where $G/U$ is a reductive group. If $G/U$ splits (e.g. if $F$ is algebraically closed), then it is rational by Proposition 1.4. Since $U$ and $G/U$ are split over an algebraically closed field, we get

**Proposition 1.6.** ([9]) Any connected algebraic group over an algebraically closed field is rational.

Denote by $T$ the variety of maximal tori in $G$. It is proved in [9, Prop.3] (in characteristic zero), [12, Exp.XIV, Th. 6.2] and [5, Th. 7.9] that $T$ is a rational variety and if $G$ is reductive, then the function field $F(G)$ is isomorphic to the function field of the generic maximal torus $T$ in $G$ defined over $F(T)$. Hence we get

**Proposition 1.7.** If all maximal tori in a reductive group $G$ are rational, then $G$ is rational.

The rank of a reductive group $G$ is the dimension of a maximal torus in $G$. Since all algebraic tori of dimension at most 2 are rational ([46, Th.4.74]), we get the following

**Corollary 1.8.** Any reductive group of rank at most 2 is rational.

The first example of a non-rational algebraic group (3-dimensional torus) was constructed by Chevalley in [9]. In [37] Rosenlicht has given an example of a 1-dimensional unipotent group $G$ over an infinite field with finite group of rational points. Clearly, $G$ is not rational.

Serre has found an example of a semisimple group which does not satisfy the weak approximation property (and hence is not rational, see [39]). Note that the weak approximation holds for simply connected and adjoint groups over global fields (see [35]). The rationality problem over local fields was studied in [8].

An example of a simply connected semisimple non-rational group was given by Platonov in [32]. The first example of adjoint semisimple non-rational group was constructed in [23].

2. $R$-equivalence for algebraic groups

In this section we define an important birational invariant of an algebraic group: the group of $R$-equivalence classes. The notion of $R$-equivalence was introduced by Manin in [18] and studied for algebraic groups by Colliot-Thélène and Sansuc in [10].

Let $F$ be a field. Denote by $\mathcal{O}$ the semilocal subring in the field of rational functions $F(t)$ consisting of all functions defined at points $t = 0$ and $t = 1$. Evaluation at these points gives two $F$-algebra homomorphisms $e_i : \mathcal{O} \to F$, $i = 0, 1$.

Let $X$ be an algebraic variety defined over a field $F$. For any element $x(t) \in X(\mathcal{O})$ and $i = 0, 1$ denote by $x(i)$ the value of $x(t)$ at $i$, i.e. the image of $x(t)$ under the map $X(e_i) : X(\mathcal{O}) \to X(F)$.
Let $G$ be an algebraic group defined over $F$. An element $g \in G(F)$ is called \textit{R-trivial} if there is $g(t) \in G(O)$ such that $g(0) = 1$ and $g(1) = g$. The set of all $R$-trivial elements in $G(F)$ we denote by $RG(F)$.

**Lemma 2.1.** $RG(F)$ is a normal subgroup in $G(F)$.

\textbf{Proof.} Let $g_1, g_2 \in RG(F)$. Choose $g_{1,2}(t) \in G(O)$ such that $g_{1,2}(0) = 1$, $g_{1,2}(1) = g_{1,2}$ and consider $g(t) = g_1(t) \cdot g_2(t)^{-1} \in G(O)$. Then $g(0) = 1$ and $g(1) = g_1 g_2^{-1}$, hence $g_1 g_2^{-1} \in RG(F)$, i.e. $RG(F)$ is a subgroup in $G(F)$. Let $g \in G(F)$ and $g' \in RG(F)$. Choose $g'(t) \in G(O)$ such that $g'(0) = 1$ and $g'(1) = g'$ and consider $g''(t) = g \cdot g'(t) \cdot g^{-1} \in G(O)$. Then $g''(0) = 1$ and $g''(1) = gg'g^{-1}$, hence $gg'g^{-1} \in RG(F)$ and $RG(F)$ is a normal subgroup in $G(F)$. $\square$

We simply denote the factor-group $G(F)/RG(F)$ by $G(F)/R$ and call it the \textit{group of $R$-equivalence classes}. An algebraic group $G$, defined over $F$, is called \textit{R-trivial} if $G(E)/R = 1$ for any field extension $E/F$.

The relation between $R$-triviality and stable rationality of an algebraic group is given by the following

**Proposition 2.2.** ([10]) If a connected algebraic group $G$, defined over a field $F$, is stably rational, then the group $G$ is $R$-trivial.

\textbf{Proof.} If $G$ is stably rational over $F$, then it is stably rational over an arbitrary field extension of $F$. Hence, it is sufficient to show that $G(F)/R = 1$. Assume first that $G$ is rational. The case of a finite field $F$ is considered in [10, Corollary 6]. Over an infinite field there is an open subset $U \subset G$ with $U(F) \neq \emptyset$ isomorphic to an open subset $V$ in some affine space $A^n_F$. Let $x \in G(F)$ be any rational point. Consider two translations $U_1$ and $U_2$ of $U$ containing points $x$ and $1$ respectively. Choose any rational point $y$ in the intersection $U_1 \cap U_2$. Translating the intersections with $V$ of two appropriate straight lines from $A^n_F$ to $U_1$ and $U_2$ we get elements $g(t), h(t) \in G(O)$ such that $g(0) = h(0) = y$, $g(1) = x$ and $h(1) = 1$. Then for $f(t) = g(t)h(t)^{-1}$ we have: $f(0) = 1$ and $f(1) = x$, hence $x \in RG(F)$.

If $G$ is stably rational then $G \times_F A^n_F$ is rational for some $n \in \mathbb{N}$. Since $A^n_F$ is a rational algebraic group, it follows from the first part of the proof that $G(F)/R = G(F)/R \times A^n_F(F)/R = (G \times_F A^n_F)(F)/R = 1$. $\square$

Let $\alpha : G \to H$ be an algebraic group homomorphism over a field $F$. It induces homomorphisms

$$\alpha_F : G(F) \to H(F) \quad \text{and} \quad \alpha_O : G(O) \to H(O).$$

Let $g \in RG(F)$, so that there exists $g(t) \in G(O)$ such that $g(0) = 1$ and $g(1) = g$. Denote by $h(t)$ the element $\alpha_O(g(t)) \in H(O)$. Clearly, $h(0) = \alpha_F(g(0)) = \alpha_F(1) = 1$ and $h(1) = \alpha_F(g(1)) = \alpha_F(g)$, hence, $\alpha_F(g) \in RH(F)$. In other words, $f_F$ takes $RG(F)$ to $RH(F)$ and therefore induces a group homomorphism $G(F)/R \to H(F)/R$. Thus, the correspondence $G \mapsto G(F)/R$ induces a functor from the category of algebraic groups over $F$ to the category of (abstract) groups.
3. Stably birational classification of algebraic tori

Let $L/F$ be a finite Galois field extension with the Galois group $\Pi$, $T$ an algebraic torus, defined over a field $F$ and split by $L$. Denote by $T^*$ the $\Pi$-module of characters

$$\text{Hom}_L(T_L, G_{m,L}).$$

It is a free abelian group of finite rank.

A $\Pi$-module $M$, being a free abelian group of finite rank, we call a $\Pi$-lattice. A $\Pi$-lattice $M$ is a permutation module if there there is a $\mathbb{Z}$-base of $M$ which is invariant under $\Pi$. A torus $T$ is quasi-trivial iff $T^*$ is a permutation $\Pi$-module. Two $\Pi$-modules $M$ and $N$ are called stably isomorphic, if $M \oplus P_1 \simeq N \oplus P_2$ for some permutation $\Pi$-modules $P_1, P_2$. The monoid (with respect to the direct sum) of stable isomorphism classes of $\Pi$-lattices we denote by $C(\Pi)$.

Let $M$ be a $\Pi$-lattice. An exact sequence of $\Pi$-lattices

$$0 \to M \to P \to N \to 0$$

is called a flasque resolution of $M$ if $P$ is a permutation $\Pi$-module and $H^{-1}(\Pi', N) = 0$ for any subgroup $\Pi' \subset \Pi$. A flasque resolution always exists and the stable isomorphism class of the module $N$ in $C(\Pi)$ is uniquely determined by $M$ (see [10]). We denote it by $p(M)$.

**Theorem 3.1.** (Colliot-Thélène – Sansuc [10], Voskresenskiî [46]) Let $T_1$ and $T_2$ be two algebraic tori split by $L/F$. They are stably birationally isomorphic iff $p(T_1^*) = p(T_2^*) \in C(\Pi)$. A torus $T$, split by $L/F$, is stably rational iff $p(T^*) = 0 \in C(\Pi)$.

The group of $R$-equivalence classes of a torus $T$ can be also computed by using a flasque resolution

$$0 \to T^* \to P \to N \to 0$$

of the $\Pi$-module of characters $T^*$.

**Proposition 3.2.** (Colliot-Thélène – Sansuc [10]) The group of $R$-equivalence classes $T(F)/R$ is naturally isomorphic to $H^1(\Pi, S(L))$, where $S$ is a torus, such that $S^* \simeq N$.

The notion of $R$-triviality for algebraic tori is very close to the stable rationality. It is interesting whether a similar statement holds for reductive groups.

**Proposition 3.3.** (Colliot-Thélène – Sansuc [10]) Let $T$ be an algebraic tori split by a finite Galois extension $L/F$, $\Pi = \text{Gal}(L/F)$. Then the following conditions are equivalent

1. $T$ is $R$-trivial;
2. There is a torus $S$, such that $T \times S$ is a rational torus;
3. $p(T^*)$ is an invertible element in $C(\Pi)$.
4. Semisimple classical groups

Let $F$ be any field of characteristic different from 2, $Z$ be either a field $F$ or a quadratic etale extension of $F$ (not necessarily a field). Consider an Azumaya algebra $A$ over $Z$ with involution $\sigma$ such that $F$ coincides with the subfield of $\sigma$-invariant elements in $Z$. Of course, $A$ is a central simple $Z$-algebra if $Z$ is a field. By Wedderburn theorem, $A \simeq \text{End}_D(V)$ for some skewfield $D$ and a vector space $V$ over $D$. There is an $\varepsilon$-hermitian form $h$ on $V$ over $D$ with respect to some involution on $D$, such that $\sigma$ is adjoint to $h$, i.e.

$$h(a(v), u) = h(v, \sigma(a)(u))$$

for all $a \in A$ and $u, v \in V$ [14]. The involution $\sigma$ is called isotropic (resp. hyperbolic), if the hermitian form $h$ is isotropic.

An element $a \in A^\times$ is called a similitude if $\sigma(a)a \in F^\times$. Denote by $\text{Sim}(A, \sigma)$ the group of all similitudes. For any $a \in \text{Sim}(A, \sigma)$ the element $\mu(a) = \sigma(a)a$ in $F^\times$ is called the similarity factor of the similitude $a$. Moreover, we have a group homomorphism

$$\mu : \text{Sim}(A, \sigma) \to F^\times.$$ 

The image of $\mu$ we denote by $G(A, \sigma)$.

An element $a \in A^\times$ is called an isometry if $\sigma(a)a = 1$. The group of all isometries $\text{Iso}(A, \sigma)$ coincides with the kernel of $\mu$.

We consider the groups $\text{Sim}(A, \sigma)$ and $\text{Iso}(A, \sigma)$ as the groups of $F$-points of the corresponding algebraic groups $\text{Sim}(A, \sigma)$ and $\text{Iso}(A, \sigma)$. The latter algebraic group is the kernel of the algebraic group homomorphism

$$\mu : \text{Sim}(A, \sigma) \to G_{m,F}.$$ 

Since $Z^\times$ is in the center of $\text{Sim}(A, \sigma)$, it follows that the torus $R_{Z/F}(G_{m,Z})$ is a central subgroup in $\text{Sim}(A, \sigma)$. The group of projective similitudes is the factor-group $\text{Sim}(A, \sigma)/R_{Z/F}(G_{m,Z})$ which we denote by $\text{PSim}(A, \sigma)$. By Hilbert’s theorem 90, the group of $F$-points of $\text{PSim}(A, \sigma)$ equals

$$\text{PSim}(A, \sigma)(F) = \text{Sim}(A, \sigma)/Z^\times.$$ 

For any $a \in A^\times$ the inner automorphism $\text{Int}(a)$ of $A$ commutes with $\sigma$ iff $a \in \text{Sim}(A, \sigma)$. By Skolem-Noether theorem, the correspondence $a \mapsto \text{Int}(a)$ induces an isomorphism of the group $\text{Sim}(A, \sigma)/Z^\times$ and the group $\text{Aut}_Z(A, \sigma)$ of $Z$-automorphisms of $A$ commuting with $\sigma$.

Denote by $\text{Sim}_+(A, \sigma)$, $\text{PSim}_+(A, \sigma)$, $\text{Aut}_+(A, \sigma)$ and $\text{Iso}_+(A, \sigma)$ the connected components of the identity in the corresponding algebraic groups. We have the canonical isomorphism

$$\text{PSim}_+(A, \sigma) \simeq \text{Aut}_+(A, \sigma).$$ 

A relation between classical semisimple groups and algebras with involutions is given by the following classical
Theorem 4.1. (Weil, [49]) Any adjoint absolutely simple classical algebraic group, defined over $F$, is isomorphic to $\text{PSim}_+(A, \sigma)$ for a suitable algebra with involution $(A, \sigma)$.

Denote the subgroup $\mu_F(\text{Sim}_+(A, \sigma)(F)) \subset F^\times$ by $G_+(A, \sigma)$. In the case, when $A$ splits, $\sigma$ is adjoint to a symmetric bilinear form (quadratic form) $q$, $G_+(A, \sigma) = G(A, \sigma) = G(q)$ is the group of similarity factors of $q$.

Denote by $\text{Hyp}(A, \sigma)$ the subgroup in $F^\times$ generated by the norms in all finite extensions $E/F$, such that $\sigma_E$ is a hyperbolic involution.

The following Theorem computes the group $\text{PSim}_+(A, \sigma)$ modulo $R$-equivalence.

Theorem 4.2. ([23]) There is a natural isomorphism

$$\text{PSim}_+(A, \sigma)(F)/R \cong G_+(A, \sigma)/(NZ^\times \cdot \text{Hyp}(A, \sigma)).$$

The group of isometries behaves much better.

Lemma 4.3. ([49]) The group $\text{Iso}_+(A, \sigma)$ is rational.

Proof. The Cayley transformation $a \mapsto (1 - a)/(1 + a)$ establishes a birational isomorphism between $G = \text{Iso}_+(A, \sigma)$ and the affine space of all skew-symmetric elements in $A$ with respect to $\sigma$ (the Lie algebra of $G$).

Remark 4.4 In the case $A$ splits, $A = \text{End}(V)$, and $\sigma$ is adjoint to a quadratic form $q$, the group $\text{Iso}_+(A, \sigma)$ is isomorphic to the special orthogonal group $\text{O}_+(q)$, so that the latter group is rational.

At the end of this section we prove the following useful technical result.

Proposition 4.5. Let $1 \to H_i \to G_i \xrightarrow{f_i} T \to 1$, $i = 1, 2$, be two exact sequences of connected algebraic groups. Assume that for any field extension $E/F$ the images of $f_1^E$ and $f_2^E$ coincide in $T(E)$. Then the groups $H_1 \times G_2$ and $H_2 \times G_1$ are birationally isomorphic.

Proof. Let $G = G_1 \times_T G_2$, $E = F(G_1)$ and $x \in G_1(E)$ be the generic point. By assumption, $f_1^E(x)$ belongs to the image of $f_2^E$. By the universal property of the fibred product, $x$ belongs to the image of $G(E) \to G_1(E)$, i.e. the exact sequence $1 \to H_2 \to G \to G_1 \to 1$ splits rationally and $G$ is birationally isomorphic to $H_2 \times G_1$. Analogously, $G$ is birationally isomorphic to $H_1 \times G_2$.

Corollary 4.6. Let $G$ be a connected stably rational algebraic group over a field $F$ and $\alpha : G \to G_{m, F}$ be a homomorphism defined over $F$. Assume that

1. The kernel of $\alpha$ is a connected stably rational algebraic group defined over $F$.
2. For any field extension $E/F$ the image of $\alpha_E$ in $E^\times$ equals $G_+(A_E, \sigma_E)$.

Then the variety of the group $\text{PSim}_+(A, \sigma)$ is stably rational.

Proof. We can apply Proposition 4.5 to the following exact sequences

$$1 \to \ker(\alpha) \to G \to G_{m, F} \to 1, \quad 1 \to \text{Iso}_+(A, \sigma) \to \text{Sim}_+(A, \sigma) \xrightarrow{\mu} G_{m, F} \to 1.$$
and conclude that $\text{Sim}_+(A, \sigma)$ is stably rational by Lemma 4.3. By Lemma 1.1 and Hilbert’s theorem 90, the exact sequence

$$1 \to R_{Z/F}(G_{m,Z}) \to \text{Sim}_+(A, \sigma) \to \text{PSim}_+(A, \sigma) \to 1$$

splits rationally, hence the claim. \hfill \Box

5. Type $A_{n-1}$

An arbitrary semisimple simply connected (resp. adjoint) algebraic group over a field $F$ is isomorphic to a direct product of several groups of the type $G_1 = R_{E/F}(G)$, where $E/F$ is a finite separable field extension and $G$ is absolutely simple simply connected (resp. adjoint) adjoint algebraic group over $E ([43, 38, 3.1.2])$. Since $R$-equivalence commutes with direct products and $G_1(F)/R = G(E)/R$, the computation of the group of $R$-equivalence classes for a semisimple simply connected (resp. adjoint) algebraic group reduces to the case of absolutely simple algebraic groups of the same type.

Below we compute the group of $R$-equivalence classes for all classical absolutely simple simply connected and adjoint groups of types $A$, $B$, $C$ and $D$. We follow the classification of adjoint groups given in [49], [44] and [15]. In this section we consider groups of type $A$ over a field $F$ of arbitrary characteristic.

5.1 Simply connected groups

An arbitrary absolutely simple simply connected algebraic group of the type $A_{n-1}$ is isomorphic to the special unitary group $\text{SU}(A, \sigma)$, where $A$ is an Azumaya algebra of degree $n$ over an etale quadratic extension $Z$ of $F$ and $\sigma$ is an involution of the second kind trivial on $F$. The group of $F$-points of $\text{SU}(A, \sigma)$ consists of all $aG_i$, such that $(a)a - 1$ and $Nrd(a) - 1$.

Consider first the case when $Z$ splits, $Z = F \times F$. Then $A$ is isomorphic to $B \times B^{\text{op}}$ with the switch involution, where $B$ is a central simple algebra of degree $n$ over $F$. It is easy to see that the map $b \mapsto (b, (b^{-1})^{\text{op}})$ gives rise to an isomorphism between $\text{SL}_1(B)$ and $\text{SU}(A, \sigma)$. The group of $R$-equivalence classes is computed in the following Proposition. Here comes the algebraic $K$-theory.

**Proposition 5.1.** ([46]) $\text{SL}_1(B)/R \simeq SK_1(B) = \text{SL}_1(B)/[B^\times, B^\times]$. \hfill \Box

**Example 5.2** If $\text{ind}(B)$ is square-free, then $SK_1(B) = 1 ([48])$. Hence the group $\text{SL}_1(B)$ is $R$-trivial in this case. It is not known whether these groups are rational.

By Wedderburn theorem $B \simeq M_n(D)$ for some skewfield $D$ over $F$. Then the group $\text{SL}_1(B)$ is isomorphic to $\text{SL}_n(D)$. Let $U$ (resp. $U^-$) be the subgroup of upper (resp. lower) strictly triangular matrices, $T$ be the maximal torus of diagonal matrices. Since $\text{SL}_1(B)$ is birationally isomorphic to $U^- \times T \times U$ and $T \simeq \text{SL}_1(D) \times \text{GL}_{n-1}(D)$, it follows that $\text{SL}_1(B)$ is birationally isomorphic to $\text{SL}_1(D) \times A^m$ for some $m$. Hence, by Corollary 1.8, we get

**Proposition 5.3.** If $\text{ind}(B) \leq 3$, then $\text{SL}_1(B)$ is a rational group. \hfill \Box

Suslin has conjectured ([41]) that if $\text{ind}(B)$ is not square-free, then the group $\text{SL}_1(B)$ is not $R$-trivial and hence is not stably rational. The following statement confirms this conjecture in a special case.
Theorem 5.4. ([19]) If \( \text{ind}(B) \) is divisible by 4, then the group \( SL_1(B) \) is not \( R \)-trivial and hence is not stably rational.

Remark 5.5 The proof of the theorem is based on Rost’s computation of the group \( SK_1 \) for a biquaternion algebra, which, in its turn, uses in particular Swan’s computation of the higher algebraic \( K \)-theory of a projective quadric hypersurface.

Remark 5.6 Examples of central simple algebras \( B \) of arbitrary index divisible by 4 and hence of non-rational groups \( SL_1(B) \) exist over any number field. Note that by [48], the group of \( R \)-equivalence classes \( SL_1(B)/R \) is trivial in this case.

Now consider the case when \( Z \) is a quadratic field extension of \( F \). Denote by \( \Sigma' \) the group of all \( a \in A^\times \), such that \( \text{Nrd}(a) \in F^\times \). Any invertible \( \sigma \)-symmetric element \( a \in A^\times \) belongs to \( \Sigma' \). Denote by \( \Sigma \) the subgroup in \( \Sigma' \), generated by invertible \( \sigma \)-symmetric elements in \( A \), so that \( \Sigma \subset \Sigma' \). The group \( \Sigma \) does not depend on the choice of involution of the second kind on \( A \).

Let \( a \in SU(A, \sigma) \). Choose an element \( z \in Z \) with the \((Z/F)\)-trace 1 (one can take \( z = 1/2 \) if \( \text{char}(F) \neq 2 \)), such that the element \( za + \sigma(z) \) is invertible. Since for any \( a \) in \( SU(A, \sigma) \) one has \( \sigma(a)a = 1 \) and \( \text{Nrd}(a) = 1 \), it follows that

\[
\sigma(za + \sigma(z)) \cdot a = za + \sigma(z)
\]

and hence \( za + \sigma(z) \in \Sigma' \).

Consider the map

\[
\varphi : SU(A, \sigma) \to \Sigma'/\Sigma,
\]

defined by the formula \( \varphi(a) = (za + \sigma(z)) \cdot \Sigma \).

Theorem 5.7. ([7, Th. 5.4]) The map \( \varphi \) is a well-defined homomorphism, which induces an isomorphism \( SU(A, \sigma)/R \cong \Sigma'/\Sigma \).

By [50], the group \( \Sigma'/\Sigma \) is trivial if \( \text{ind}(A) \) is square-free. Thus, we obtain

Corollary 5.8. If \( \text{ind}(A) \) is square-free, then the group \( SU(A, \sigma) \) is \( R \)-trivial.

Let \( D \) be a skewfield over \( Z \) Brauer equivalent to \( A \). One can prove the following

Theorem 5.9. ([7, Th. 5.4]) There is an involution \( \tau \) of the second kind on \( D \), such that \( SU(A, \sigma) \) is birationally isomorphic to \( SU(D, \tau) \times A^m_\mathbb{P} \) for some \( m \).

Corollary 1.8 then gives

Corollary 5.10. ([7, Cor. 5.5]) Let \( A \) be a central simple \( Z \)-algebra with an involution \( \sigma \) of the second kind trivial on \( F \). If \( \text{ind}(A) \leq 3 \), then the variety of the group \( SU(A, \sigma) \) is rational.

Remark 5.11 Since \( SU(A, \sigma) \otimes_F Z \cong SL_1(A) \), it follows from Theorem 5.4 that the group \( SU(A, \sigma) \) is not rational if \( \text{ind}(A) \) is divisible by 4.

Let \( T = \ker(N_{Z/F} : R_{Z/F}(G_{m, Z}) \to G_{m, F}) \) be the norm 1 torus in the quadratic extension \( Z/F \). The group \( SU(A, \sigma) \) is the kernel of the reduced norm
homomorphism \( \text{Nrd} : \text{U}(A, \sigma) \to T \), where \( \text{U}(A, \sigma) = \text{Iso}(A, \sigma) \) is the unitary group. By [22, Prop. 6.1], for any field extension \( E/F \), the image of this homomorphism on the groups of \( E \)-points depends only on the Brauer class of \( A \). Applying Lemma 4.3 and Proposition 4.5 to exact sequences

\[
1 \to \text{SU}(A, \sigma) \to \text{U}(A, \sigma) \to T \to 1, \quad 1 \to \text{SU}(A', \sigma') \to \text{U}(A', \sigma') \to T \to 1,
\]

we get the following

**Corollary 5.12**. ([7, Cor. 5.7]) Let \( (A, \sigma) \) and \( (A', \sigma') \) be central simple \( \mathbb{Z} \)-algebras with involutions of the second kind. If \( A \) and \( A' \) are Brauer equivalent algebras, then the groups \( \text{SU}(A, \sigma) \) and \( \text{SU}(A', \sigma') \) are stably birationally isomorphic.

**Remark 5.13** Corollary shows that the stable birational type of the group \( \text{SU}(A, \sigma) \) depends only on the Brauer class of \( A \) and does not depend on the involution \( \sigma \).

### 5.2 Adjoint groups

An arbitrary absolutely simple adjoint algebraic group of the type \( A_{n-1} \) is isomorphic to the connected group \( \text{PSim}(A, \sigma) \), where \( A \) is an Azumaya algebra of degree \( n \) over an etale quadratic extension \( Z \) of \( F \) and \( \sigma \) is an involution of the second kind trivial on \( F \).

Consider first the case when \( Z \) splits, i.e. \( A \) is isomorphic to \( B \times B^{\text{op}} \) with the switch involution, where \( B \) is a central simple algebra of degree \( n \) over \( F \). It is easy to see that the map \( b \mapsto (b, (b^{-1})^{\text{op}}) \) gives rise to an isomorphism of \( \text{PGL}_1(B) \) and \( \text{PSim}(A, \sigma) \). The group \( \text{PGL}_1(B) \) embeds as an open subset in the projective space \( \mathbb{P}(B) \), and hence is rational.

Consider now the case when \( Z \) is a field. The group \( \text{PSim}(A, \sigma) \) is called the *projective unitary group* and is denoted by \( \text{PGU}(A, \sigma) \). The involution \( \sigma_Z \) is hyperbolic since \( Z \otimes_F Z \) splits, hence \( NZ^\times \subset \text{Hyp}(A, \sigma) \). It follows from Theorem 4.2 that

\[
\text{PGU}(A, \sigma)(F)/R = G(A, \sigma)/\text{Hyp}(A, \sigma).
\]

Consider the following particular cases:

**Case 1:** \( n = 2 \).

In this case \( A \) is a quaternion algebra over \( Z \). Denote by \( (a \mapsto \bar{a}) \) the canonical involution on \( A \) and consider the quaternion \( F \)-subalgebra

\[
Q = \{ a \in A \text{ such that } \sigma a = \bar{a} \}
\]

in \( A \). Then \( \text{PGU}(A, \sigma) \simeq \text{PGL}_1(Q) \) is a rational algebraic group (see [15]).

**Case 2:** \( n \) is odd.

In this case \( G(A, \sigma) = \text{Hyp}(A, \sigma) = NZ^\times ([23]) \), in particular

\[
\text{PGU}(A, \sigma)(F)/R = 1.
\]

This reflects the fact that the algebraic group \( \text{PGU}(A, \sigma) \) is rational if \( n \) is odd ([47, Cor. of Th. 8]). This statement is proved by showing that all maximal tori in \( \text{PGU}(A, \sigma) \) are rational (see Proposition 1.7).

**Case 3:** \( n \) is even.

Denote by \( D(A, \sigma) \) the discriminant algebra of \( (A, \sigma) \) (see [15], [25]).
**Theorem 5.14.** Assume that

1. \( \text{deg}(A) \) is divisible by 4;
2. \( \text{ind}(A) \) is even;
3. The algebra \( D(A, \sigma) \) is not split.

Then the group \( \text{PGU}(A, \sigma) \) is not \( R \)-trivial and hence is not stably rational.

**Proof.** We will show that there is a field extension \( E/F \) such that

\[
\text{PGU}(A_E, \sigma_E)/R \neq 1.
\]

Let \( X = \text{SB}(A, 2) \) be the generalized Severi-Brauer variety of right ideals in \( A \) of dimension \( 2 \cdot \text{deg}(A) \) over \( Z \) ([3]) and \( X' = R_{Z/F}(X) \). The kernel of \( \text{Br}(Z) \to \text{Br}(Z(X)) \) is generated by the class of \( A \otimes_Z A \) ([3]). Since the corestriction of \( A \) in the extension \( Z/F \) is trivial ([11]), by [21, Cor. 2.12], the natural homomorphism \( \text{Br}(F) \to \text{Br}(F(X')) \) is injective, hence \( D(A, \sigma) \) is not split over \( F(X') \). Since \( X' \otimes_F Z \simeq \text{SB}(A, 2) \times \text{SB}(A^\text{op}, 2) \), by the index reduction formula (see [3], [26]), the index of \( A \times_F F(X') = A \otimes_Z Z(X') \) equals 2. Replacing the field \( F \) by \( F(X') \), we may assume that \( \text{ind}(A) = 2 \), so that \( A \simeq \text{End}_Q(V) \) for some quaternion algebra \( Q \) and a \( Q \)-vector space \( V \) of even dimension \( 2k \). The involution \( \sigma \) is adjoint to some hermitian form \( h \) on \( V \) (with respect to some involution of the second kind on \( Q \)).

Let \( Y \) be the variety over \( F \) of isotropic subspaces in \( V \) of dimension \((k - 1) \) over \( Q \). By [21, Cor. 2.8], the natural homomorphism \( \text{Br}(F) \to \text{Br}(F(Y)) \) is injective, hence \( D(A, \sigma) \) is not split over \( F(Y) \). Since \( Y \) has a rational point over \( Z \), it follows that \( A \otimes_F F(Y) = A \otimes_Z Z(Y) \) is not split. Hence, replacing \( F \) by \( F(Y) \), we may assume that \( V = V' \perp V'' \), where \( V'' \) is a hyperbolic space of dimension \( 2(k - 1) \) and \( \dim(V') = 2 \).

Consider the algebra \( A' = \text{End}_Q(V') \) of degree 4 and the involution \( \sigma' \) of the second kind on \( A' \) adjoint to the hermitian form \( h|_{V'} \) on \( V' \). By Theorem 4.2, the groups of \( R \)-equivalence classes for \( \text{PGU}(A, \sigma) \) and \( \text{PGU}(A', \sigma') \) coincide over any field extension of \( F \). The discriminant algebra \( D(A', \sigma') \), being similar to \( D(A, \sigma) \), is not trivial, hence by Theorem 9.3 in Section 9 (see also Remark 9.5), the group \( \text{PGU}(A', \sigma') \), and therefore \( \text{PGU}(A, \sigma) \), is not \( R \)-trivial.

**Remark 5.15** Theorem gives examples of non-rational adjoint groups of types \( A_{4k - 1} \) for all \( k \) over any number field.

**6. Type \( B_n \)**

We assume that characteristic of the base field \( F \) is different from 2.

**6.1 Simply connected groups**

An arbitrary absolutely simple simply connected algebraic group of the type \( B_n \) is isomorphic to the group \( \text{Spin}(q) = \text{Spin}(V, q) \), where \((V, q)\) is a quadratic form (uniquely determined up to a scalar) of dimension \( 2n + 1 \geq 3 \). In this section we collect some results for arbitrary forms (not necessarily of odd dimension).
PROPOSITION 6.1. ([33]) If \( q \) is an isotropic form, then the group \( \text{Spin}(q) \) is rational.

Proof. Since \( q \) is isotropic, there are two vectors \( u, v \in V \), such that \( v^2 = u^2 = 0 \) and \( uv + vu = 1 \) in the Clifford algebra. The image of the homomorphism

\[
\mathbb{G}_{m,F} \to \text{Spin}(q), \quad a \mapsto a^{-1}(uv + a^2vu)
\]

we denote by \( T \) and the factor-variety \( \text{Spin}(q)/T \) by \( X \). Since \( T \simeq \mathbb{G}_{m,F} \), by Hilbert’s theorem 90 and Lemma 1.1, the exact sequence

\[
1 \to T \to \text{Spin}(q) \to X \to 1
\]
splits rationally, hence \( \text{Spin}(q) \) is birationally isomorphic to \( T \times X \). The kernel \( K = \{ \pm 1 \} \) of the natural homomorphism \( \text{Spin}(q) \to O_+(q) \) belongs to \( T \) and the square endomorphism of \( T \) induces an isomorphism \( T/K \simeq T \). Hence, \( \text{Spin}(q) \) is birationally isomorphic to \( T/K \times AT \), and therefore, to the rational group \( \text{Spin}(q)/K \simeq O_+(q) \) (see Lemma 4.3). \( \square \)

For a quadratic form \( q \) denote by \( D(q) \) the set of non-zero values of \( q \).

LEMMA 6.2. Let \((V, q)\) be a quadratic form over \( F \) of dimenrsion at least 2, \((V', q')\) subform of codimension 1, \( L/F \) quadratic extension, such that \( q_L = q \otimes_F L \) is isotropic. Then \( N_{L/F}(L^X) \subset D(q) \cdot D(q') \).

Proof. We may clearly assume that \( q \) is anisotropic. Consider first the case \( \dim(V) = 2 \), i.e. \( q' = \langle b \rangle \) and \( q = \langle b, -ab \rangle \) for some \( a, b \in F^\times \). Then \( L = F(\sqrt{a}) \) and the result immediately follows.

Consider now the general case. Since \( q_L \) is isotropic, there is a 2-dimensional subspace \( U \subset V \), such that \( q|_U \) is isotropic over \( L \) ([16, Lemma VII.3.1]). If \( U \subset V' \), then \( q' \) is isotropic over \( L \) and by Knebusch’s norm principle ([16, Th. VII.5.1]),

\[
N_{L/F}(L^X) \subset D(q') \cdot D(q') \subset D(q) \cdot D(q').
\]

If \( U \) is not contained in \( V' \), then by the 2-dimensional case, applied to the subform \( q'|_{U \cap V'} \) in \( q|_U \), we have

\[
N_{L/F}(L^X) \subset D(q|_U) \cdot D(q'|_{U \cap V'}) \subset D(q) \cdot D(q'). \quad \square
\]

A Pfister form is a tensor product of binary forms representing 1. A Pfister neighbor is a form similar to a subform of a Pfister form \( f \) of dimension bigger than one half of \( \dim(f) \).

LEMMA 6.3. Let \( q \) be a Pfister neighbor of a Pfister form \( f \), \( q' \) subform in \( q \) of codimension 1. Then \( D(f) = D(q) \cdot D(q') \).

Proof. We may assume that \( f \) is anisotropic. Let \( x \in D(f) \). Then there exists a quadratic extension \( L/F \), such that \( f_L \) is isotropic and \( x \in N_{L/F}(L^X) \). Since \( q \) is a Pfister neighbor, \( q_L \) is also isotropic and by Lemma 6.2, \( x \in D(q) \cdot D(q') \), i.e. \( D(f) \subset D(q) \cdot D(q') \). The inverse inclusion follows from the fact that \( D(f) \) is a group ([16, Cor. X.1.7]). \( \square \)
**Theorem 6.4.** (Chernousov–Merkurjev–Rost) Let \( q \) be a Pfister neighbor and \( p \) a quadratic form of dimension at most 2. Then the group \( \text{Spin}(q \perp p) \) is rational.

**Proof.** In the space \( V \) of the form \( q \perp p \) choose subspaces \( V'' \subset V' \subset V \) of consecutive codimensions 1 so that \( V'' \) is contained in the space of \( q \). An element \( x \) in a certain open subset in \( \text{Spin}(q \perp p) \) can be written in the form \( x = y \cdot v \cdot (v')^{-1} \), where \( y \) belongs to the Clifford group of \( V'' \) and \( v' \in V', v \in V \), so that the class \([y]\) in \( \mathbf{O}_+(V'') \) is uniquely determined. We get a rational dominant morphism

\[
\text{Spin}(q \perp p) \to \mathbf{O}_+(V''), \quad x \mapsto [y].
\]

The generic fiber of this morphism is the projective quadric in the space \( V \oplus V' \), given by the quadratic form \( g = (-q_1) \perp \text{Sn}(y) \cdot q_2 \), where \( q_2 = q \perp p \) and \( q_1 = q_2|_{V'} \). Since \( q \) is a Pfister neighbor of some Pfister form \( f \), \( \text{Sn}(y) \in D(f) \). The form \( q_1 \) contains the Pfister neighbor \( q \) and \( q_1 \) is of codimension 1 in \( q_2 \), hence, by Lemma 6.3, \( g \) is isotropic, i.e. the generic fiber is a rational quadric. It remains to notice that the group \( \mathbf{O}_+(V'') \) is rational by Lemma 4.3.

**Example 6.5** Any form \( q \) of dimension at most 5 satisfies conditions of Theorem 6.4, hence the group \( \text{Spin}(q) \) is rational in this case. Of course, this result also follows from Corollary 1.8, since the rank of the group is at most 2. Note that if \( \dim(q) = 6 \) and \( q \) is anisotropic over the discriminant quadratic extension, then \( \text{Spin}(q) \) is never stably rational by Corollary 9.2.

**Example 6.6** The canonical quadratic form \( q = (1, 1, \ldots, 1) \) is a Pfister neighbor, hence \( \text{Spin}(q) \) is rational. This result was obtained in [34].

**6.2 Adjoint groups**

An arbitrary absolutely simple adjoint algebraic group of the type \( B_n \) is isomorphic to a connected group \( \mathbf{O}_+(q) \), where \( q \) is a quadratic form (uniquely determined up to a scalar) of dimension \( 2n + 1 \). This group is known to be rational by Lemma 4.3.

**7. Type \( C_n \)**

**7.1 Simply connected groups**

An arbitrary absolutely simple simply connected algebraic group of the type \( C_n \) is isomorphic to the *symplectic group* \( \text{Sp}(A, \sigma) = \text{Iso}(A, \sigma) \) where \( A \) is a central simple algebra of degree \( 2n \) over \( F \) with a symplectic involution \( \sigma \). This group is rational by Lemma 4.3.

**7.2 Adjoint groups**

An arbitrary absolutely simple adjoint algebraic group of the type \( C_n \) is isomorphic to the *projective symplectic group* \( \text{PGSp}(A, \sigma) \), where \( A \) is a central simple algebra of degree \( 2n \) over \( F \) with a symplectic involution \( \sigma \).

Consider the following particular cases:
Case 1: \( n = 1 \).

A quaternion algebra \( A \) has the unique (standard) symplectic involution \( \sigma \), hence any automorphism of \( A \) preserves \( \sigma \) and therefore \( \text{PGSp}(A,\sigma) = \text{PGL}_1(A) \) is a rational group.

Case 2: \( n = 2 \).

The space of skew-symmetric elements of trivial reduced trace carries the 5-dimensional quadratic form \( q(a) = a^2 \in F \). The map

\[
\text{Sim}(A,\sigma) \rightarrow O_+(q), \quad a \mapsto \text{Int}(a)|_V
\]

induces an isomorphism of simple adjoint algebraic groups \( \text{PGSp}(A,\sigma) \simeq O_+(q) \) of the types \( C_2 = B_2 \) ([20, Prop. 5.4], [15]). In particular, the group \( \text{PGSp}(A,\sigma) \) is rational. Of course, the rationality of the groups in the first two cases follows also from Corollary 1.8.

Case 3: \( n \) is odd.

**Lemma 7.1.** ([23, Lemma 3]) If \( n \) is odd, then \( G(A,\sigma) = \text{Hyp}(A,\sigma) = \text{Nrd}(A^X) \).

**Proposition 7.2.** ([23, Prop. 4]) Any absolutely simple adjoint group of the type \( C_n \) with odd \( n \) is stably rational and hence is \( R \)-trivial.

**Proof.** We apply Corollary 4.6 to the rational algebraic group \( G = \text{GL}(Q) \) and to the reduced norm homomorphism \( \text{Nrd} = \alpha : G \rightarrow G_{m,F} \). The kernel of \( \alpha \) is the rational algebraic group \( \text{SL}_1(Q) \) (affine quadric with a rational point) and, by Lemma 7.1, \( G(A_E,\sigma_E) = \text{Nrd}(A_E^X) = \text{im} \alpha_E \) for any field extension \( E/F \).

**8. Type \( D_n \)**

We assume that \( \text{char}(F) \neq 2 \).

**8.1 Simply connected groups**

An arbitrary absolutely simple simply connected algebraic group of the type \( D_n \) (except for some non-classical groups of the type \( D_4 \)) is isomorphic to the group \( \text{Spin}(A,\sigma) \), where \( A \) is a central simple algebra of degree \( 2n \) over \( F \) with an orthogonal involution \( \sigma \). The group \( \text{Spin}(A,\sigma) \) is the kernel of the spinor norm homomorphism

\[
\text{Sn} : \Gamma(A,\sigma) \rightarrow G_{m,F},
\]

where \( \Gamma(A,\sigma) \) is the Clifford group (see [44], [15]).

We will use the following
LEMMA 8.1. ([22, Prop. 6.2]) The image of $S_{n_F}$ on the groups of $F$-points is generated by $F \times 2$ and the norms in all finite field extensions over which $A$ splits and $\sigma$ is isotropic.

Consider the subgroup $H$ in the product $\text{GL}_1(A) \times \text{G}_{m,F}$ consisting of pairs $(a, x)$, such that $\text{Nrd}(a) = x^2$. The group $H$ is the kernel of the homomorphism

$$f : \text{GL}_1(A) \times \text{G}_{m,F} \to \text{G}_{m,F}, \quad (a, x) \mapsto \text{Nrd}(a) \cdot x^{-2}. $$

By Lemma 8.1, for any field extension $E/F$, the image of $S_{n_E}$ is contained in the image of $f_E$. Hence, by [7], there is a natural homomorphism

$$\theta : \text{Spin}(A, \sigma)/R \to H(F)/R.$$

In order to compute the group $RH(F)$ consider any symplectic involution $\tau$ on $A$ and the subgroup $\Sigma \subset A^\times$ generated by the set $S$ of $\tau$-symmetric invertible elements. The group $\Sigma$ does not depend on the choice of $\tau$. There is the pfaefian map $\text{pf} : S \to F^\times$, such that $\text{pf}(s)^2 = \text{Nrd}(s)$ for any $s \in S$ (see [15]). Denote by $\Delta$ the subgroup in $H(F)$ generated by pairs $(s, \text{pf}(s))$ for all $s \in S$. We also consider the commutant $[A^\times, A^\times]$ as a subgroup in $H(F)$, identifying an element $a$ in the commutant with the pair $(a, 1)$.

LEMMA 8.2. $RH(F) = \Delta \cdot [A^\times, A^\times]$.

Proof. Let $a, b \in A^\times$. Consider $a(t) = 1 + (a - 1)t \in (A \otimes_F \mathcal{O})^\times$ and $c(t) = [a(t), b]$. Clearly, $c(0) = [a(0), b] = [1, b] = 1$ and $c(1) = [a(1), b] = [a, b]$, i.e. the commutator $[a, b]$ is $R$-trivial and $[A^\times, A^\times] \subset RH(F)$. Similarly, if $s$ is a $\tau$-symmetric element in $A^\times$, then for the pair $h(t) = (s(t), \text{pf}(s(t)))$ in $H(\mathcal{O})$, where $s(t) = 1 + (s - 1)t$, we have $h(0) = (1, 1)$ and $h(1) = (s, \text{pf}(s))$, hence $\Delta \subset RH(F)$.

In order to prove the desired equality, consider the subgroup $\Sigma' \subset A^\times$, consisting of all $a$, such that $\text{Nrd}(a) \in F^\times 2$. There is the exact sequence

$$\{\pm 1\} \xrightarrow{\alpha} H(F)/\left(\Delta \cdot [A^\times, A^\times]\right) \xrightarrow{\beta} \Sigma'/\Sigma \to 1,$$

where $\alpha(-1) = (1, -1)$ and $\beta(a, x) = a \cdot \Sigma$. Since the latter group in the sequence does not change under purely transcendental extensions (see [28, Th. 3.2]), so does the middle one. As in [28] and [7] we conclude that $\Delta \cdot [A^\times, A^\times] = RH(F)$.

The homomorphism $\theta$ is not an isomorphism in general. For example, if $A$ splits, then $H(F) = \Delta \cdot [A^\times, A^\times]$, whereas the group $\text{Spin}(A, \sigma)$ is not necessarily $R$-trivial in this case (see Corollary 9.2). Nevertheless, there is a case when $\theta$ is an isomorphism.

PROPOSITION 8.3. If $\sigma$ is an isotropic involution, then the map

$$\theta : \text{Spin}(A, \sigma)/R \to H(F)/\left(\Delta \cdot [A^\times, A^\times]\right)$$

is an isomorphism.

Proof. By Lemma 8.1, for any field extension $E/F$ the images of $S_{n_E}$ and $f_E$ coincide. Hence, $\theta$ is an isomorphism (see [7]).

Remark 8.4 Proposition 8.3 is a generalization of [28, Th. 1.6].
8.2 Adjoint groups

An arbitrary absolutely simple adjoint algebraic group of the type $D_n$ (except for some non-classical groups of the type $D_4$) is isomorphic to the projective orthogonal group $\text{PGO}_+(A, \sigma) = \text{PSim}_+(A, \sigma)$, where $A$ is a central simple algebra of degree $2n$ over $F$ with an orthogonal involution $\sigma$. The group of $F$-points of $\text{PGO}_+(A, \sigma)$ equals $\text{Sim}_+(A, \sigma)/F^\times$, where $\text{Sim}_+(A, \sigma)$ consists of proper similitudes, i.e., similitudes $a \in \text{Sim}(A, \sigma)$, such that $\text{Nrd}(a) = \mu(a)^n$.

Denote by $C = C(A, \sigma)$ the Clifford algebra of algebra with involution $(A, \sigma)$ with the center $L/F$, being an etale quadratic extension over $F$, and by $\text{disc}(\sigma)$ the discriminant of $\sigma$ (see [15]).

Case 1: $n = 2$.

**Proposition 8.5.** ([21, Prop. 1.15], [23, Prop. 5], [15])

1. Any adjoint group of the type $D_2 = A_2$ is isomorphic to $\text{PGO}_+(A, \sigma) \simeq R_{L/F}(\text{PGL}_1(C))$ and hence is rational.
2. $\text{Hyp}(A, \sigma) = N_{L/F} \text{Nrd} C$ and $G_+(A, \sigma) = F^\times \cdot N_{L/F} \text{Nrd} C$. □

Case 2: $n = 3$.

**Proposition 8.6.** ([23, Prop. 6], [15]) Let $\text{disc}(\sigma)$ be trivial. Then $C(A, \sigma) \simeq C^+ \times C^-$ and

1. There is an isomorphism of algebraic groups $\text{PGO}_+(A, \sigma) \simeq \text{PGL}_1(C^+)$ of the types $D_3 = A_3$. In particular, the former group is rational.
2. $\text{Hyp}(A, \sigma) = \text{Nrd}(C^+)$ and $G_+(A, \sigma) = F^\times \cdot \text{Nrd}(C^+)$. □

Case 3: The algebra $A$ splits, $A = \text{End}_F(V)$.

In this case the involution $\sigma$ is adjoint to some non-degenerate quadratic form $q$ on the space $V$. The group $\text{PGO}_+(A, \sigma)$ is equal to the projective special orthogonal group $\text{PGO}_+(q)$ of the quadratic form $q$. The following statement gives examples of stably rational groups $\text{PGO}_+(q)$.

**Proposition 8.7.** ([23, Prop. 7]) If $q = f \otimes_F g$ is the tensor product of a Pfister form $f$ and a form $g$ of odd dimension over $F$, then the group $\text{PGO}_+(q)$ is stably rational. □

**Example 8.8** If $q$ is the standard form $(1, 1, \ldots, 1)$, then the group $\text{PGO}_+(q)$ is stably rational (see also [6]).

Examples of non-rational groups are given by the following

**Theorem 8.9.** ([23, Th. 2]) Let $A$ be a central simple algebra of even degree over a field $F$ with an orthogonal involution $\sigma$. If $\text{disc}(\sigma)$ is not trivial and $\text{ind} C(A, \sigma) \geq 4$, then the group $\text{PGO}_+(A, \sigma)$ is not R-trivial and hence is not stably rational. □

**Remark 8.10** One of the main ingredients of the proof is the index reduction formula (see [25], [26]) which is based on the computation of the algebraic $K$-theory of certain projective homogeneous varieties given by Panin in [30].

**Remark 8.11** Examples of groups of type $D_n$ with odd $n$, satisfying conditions of the Theorem, exist over any number field (see [23, Sec. 4]).
9. Stably birational classification of absolutely simple groups of type $A_3 = D_3$

By Corollary 1.8, all semisimple groups of rank at most 2 are rational. In this section we give complete birational classification of semisimple groups (of rank 3) of type $A_3 = D_3$.

Let $A$ be a central simple algebra of degree 6 over a field $F$ with orthogonal involution $\sigma$, $C = C(A, \sigma)$ the Clifford algebra of $(A, \sigma)$. It is a central simple algebra of degree 4 over the discriminant quadratic extension $L/F$ with the canonical involution $\tau$ of the second kind ([15]).

Consider first the simply connected case. An absolutely simple group of the type $D_3 = A_3$ is $\text{Spin}(A, \sigma) \simeq \text{SU}(C, \tau)$ (see [15]). From Theorem 5.4, Proposition 5.3, Corollary 5.10 and Remark 5.11 one deduces:

**THEOREM 9.1.** I. If $\text{disc}(\sigma)$ is trivial, i.e. $C \simeq C^+ \times C^-$, where $C^\pm$ are central simple algebras of degree 4 over $F$, then

1. If $C^+$ is not a skewfield, then $\text{Spin}(A, \sigma)$ is a rational group.
2. If $C^+$ is a skewfield, then $\text{Spin}(A, \sigma)$ is not $R$-trivial and hence is not stably rational.

II. If $\text{disc}(\sigma)$ is not trivial, then

1. If $C$ is not a skewfield, then $\text{Spin}(A, \sigma)$ is a rational group.
2. If $C$ is a skewfield, then $\text{Spin}(A, \sigma)$ is not $R$-trivial and hence is not stably rational.

**COROLLARY 9.2.** Let $q$ be a non-degenerate quadratic form of dimension 6 over $F$. Then $\text{Spin}(q)$ is not stably rational iff $q$ is anisotropic over the discriminant extension $F(\sqrt{\text{disc}(\sigma)})$ of $F$.

**Proof.** If $\text{disc}(q)$ is trivial, then $C = C_0(q) = C^+ \times C^-$ and $q$ is anisotropic iff $C^+$ is a skewfield by [2].

Assume that $\text{disc}(q)$ is not trivial. Let $L/F$ be the discriminant quadratic extension of $q$. The result follows again from the fact that $q_L$ is anisotropic iff the algebra $C = C^+(q_L)$ is a skewfield.

Now consider the adjoint case. An absolutely simple group of the type $D_3 = A_3$ is $\text{PGO}_+(A, \sigma) = \text{PGU}(C, \tau)$.

**THEOREM 9.3.** ([23, Th. 3]) I. If $\text{disc}(\sigma)$ is trivial, then the group $\text{PGO}_+(A, \sigma)$ is rational.

II. If $\text{disc}(\sigma)$ is not trivial, then

1. If $C$ splits, or if $\text{ind} C = 2$ and $A$ splits, then the group $\text{PGO}_+(A, \sigma)$ is stably rational.
2. If $\text{ind} C = 4$, or if $\text{ind} C = 2$ and $A$ is not split, then the group $\text{PSim}_+(A, \sigma)$ is not $R$-trivial and hence is not stably rational.

**COROLLARY 9.4.** Let $q$ be a non-degenerate quadratic form of dimension 6 over $F$. Then $\text{PGO}_+(q)$ is not stably rational iff the discriminant of $q$ is not trivial and $q$ is anisotropic over the discriminant extension $F(\sqrt{\text{disc}(\sigma)})$ of $F$.

**Remark 9.5** The algebra $A$ can be recovered from $(C, \tau)$ as the discriminant algebra $D(C, \tau)$ (see [15]).
Finally, let $G$ be an absolutely simple group of type $D_3$, which is neither simply connected nor adjoint. Then $G = \text{Iso}_+(A, \sigma)$ is a rational group by Lemma 4.3.

10. Algebraic groups

In this section we collect definitions and preliminary results needed in the second part of the paper. We give an algebraic definition of the fundamental group $\pi_1(G)$ for an algebraic group $G$.

10.1 Fundamental group

Let $G$ be an algebraic group defined over an algebraically closed field $F$. A loop in $G$ is a group homomorphism (co-character) $p : G_{m,F} \to G$. Two loops $p$ and $p'$ are called conjugate if there is $g \in G(F)$ such that $p'(t) = g \cdot p(t) \cdot g^{-1}$ for all $t$ in $G_{m,F}$. A loop $p$ is called contractible if $p$ extends to a group homomorphism $\text{SL}_2(F) \to G$ (we identify $G_{m,F}$ with the maximal torus of diagonal matrices in $\text{SL}_2(F)$ by $t \mapsto \text{diag}(t, t^{-1})$). If $p$ and $p'$ are two loops with commuting images, then the product $pp'$ given by $(pp')(t) = p(t)p'(t)$ is also a loop.

The fundamental group $\pi_1(G)$ of the group $G$ is the group defined by generators and relations as follows. Generators are loops in $G$. For a loop $p$ we denote by $[p]$ the corresponding generator in $\pi_1(G)$. The relations are the following:

1. $[p] = [p']$ if $p$ and $p'$ are conjugate;
2. $[p] = 1$ if $p$ is contractible;
3. $[pp'] = [p] \cdot [p']$ if the images of $p$ and $p'$ commute.

Example 10.1 Let $G$ be a commutative algebraic group. Since conjugate loops are equal and contractible loops are trivial ($\text{SL}_2(F)$ coincides with its commutant), the group $\pi_1(G)$ equals the group $G_* = \text{Hom}(G_{m,F}, G)$ of co-characters.

Now let $G$ be an algebraic group over an arbitrary field $F$. The fundamental group $\pi_1(G)$ of $G$ is, by definition, the group $\pi_1(G_{F_{\text{alg}}})$, where $F_{\text{alg}}$ is an algebraic closure of $F$.

A group homomorphism $G \to G'$ clearly induces a homomorphism of fundamental groups $\pi_1(G) \to \pi_1(G')$.

Proposition 10.2. Let $G$ be a reductive group over $F$. Then

1. The fundamental group $\pi_1(G)$ of a reductive group $G$ is an abelian finitely generated group.
2. $\pi_1(G)$ is finite iff $G$ is a semisimple group.
3. $\pi_1(G)$ is trivial iff $G$ is a simply connected semisimple group.
4. If $G'$ is the commutant of $G$, then $\pi_1(G') = \pi_1(G)_{\text{tors}}$.
5. $\pi_1(G)$ is torsion free iff $G'$ is a simply connected group. \hfill $\square$

Remark 10.3 If $G$ is an algebraic group over the field of complex numbers, then $\pi_1(G)$ coincides with the ordinary fundamental group of the corresponding Lie group [1, Th.5.47]. The algebraic definition of the fundamental group, we present here, is not based on the choice of a maximal torus. That makes evident the functorial properties of the fundamental group.
10.2 Factorial groups
An algebraic group $G$ over a field $F$ is called factorial if for any finite field extension $E/F$ the Picard group $\text{Pic}(G_E)$ is trivial.

**Example 10.4** Let $T$ be an algebraic torus over $F$, $L/F$ be a finite Galois splitting field, $\Pi = \text{Gal}(L/F)$. The torus $T$ is factorial iff $T$ is coflasque, i.e. $H^1(\Pi', T^*) = 0$ for any subgroup $\Pi' \subset \Pi$ (see [10]). A quasi-trivial torus is factorial.

**Example 10.5** Let $G$ be a semisimple algebraic group over $F$, $\pi : \tilde{G} \to G$ be the universal covering, $C = \ker(\pi)$. By [38, Lemme 6.9], there is a natural isomorphism $\text{Pic}(G) \simeq C^*$. Hence $G$ is factorial iff $G$ is simply connected.

The following Proposition reduces the case of a reductive group to these two examples.

**Proposition 10.6.** Let $G$ be a reductive group over a field $F$, $G'$ be the commutant of $G$. Then $G$ is factorial iff $G'$ and the torus $G/G'$ are factorial, i.e. $G'$ is simply connected and the torus $G/G'$ is coflasque. The group $\pi_1(G)$ of a factorial reductive group $G$ is torsion free.

10.3 Reductive groups of inner type
Let $G$ and $H$ be algebraic groups over $F$. The group $G$ is called a twisted form of $H$ if $G_{\text{sep}}$ and $H_{\text{sep}}$ are isomorphic over $F_{\text{sep}}$. Let $G$ be a reductive group over $F$, $G^d$ be its split twisted form. Denote by $\overline{G}^d$ the corresponding split adjoint group $G^d/\text{Center}(G^d)$. The group $G$ is of inner type if the class of the 1-cocycles in $H^1(F, \text{Aut}(G^d_{\text{sep}}))$, corresponding to $G$, belongs to the image of the map

$$H^1(F, \overline{G}^d(F_{\text{sep}})) \to H^1(F, \text{Aut}(G^d_{\text{sep}}))$$

induced by the conjugation homomorphism

$$\text{Int} : \overline{G}^d(F_{\text{sep}}) \to \text{Aut}(G^d_{\text{sep}}).$$

A quasi-split reductive group of inner type is split.

**Proposition 10.7.** Let $G$ be a reductive group, $G'$ be the commutant of $G$. Then $G$ is of inner type iff $G'$ is of inner type and the torus $G/G'$ splits.

**Corollary 10.8.** Let $G$ be a reductive group of inner type. Then $G$ is factorial iff $\pi_1(G)$ is torsion free.

**Example 10.9** The group $\text{GL}_1(A)$ for a central simple $F$-algebra $A$ is a factorial reductive group over $F$ of inner type.

10.4 Category of representations
Let $G$ be an algebraic group defined over a field $F$. Denote by $\text{Rep}(G)$ the category of finite dimensional linear representations of $G$. It is an abelian category in which any object has finite length. Hence by [36, Th.4, Cor.1]

$$K_n\left(\text{Rep}(G)\right) = \prod_{V \text{ irreducible}} K_n\left(\text{End}_G(V)^{op}\right).$$
In particular, $K_0(\text{Rep}(G))$ is a free abelian group freely generated by the classes of irreducible representations. We denote this group by $R(G)$. It has a ring structure with respect to the tensor product.

There is the rank homomorphism of rings $\text{rk} : R(G) \to \mathbb{Z}$, taking the class of a $G$-module $V$ to $\dim V$. We denote its kernel by $J(G)$.

A homomorphism $f : H \to G$ of algebraic groups over $F$ induces a ring homomorphism $f^* : R(G) \to R(H)$. Thus, we can consider $R(H)$ as a $R(G)$-module. If $f$ is the embedding of a subgroup $H$ to $G$, the homomorphism $f$ is called the restriction map.

**Example 10.10** For a split solvable group $B$ over $F$, $R(B) = \mathbb{Z}[B^*]$ is a Laurent polynomial ring over $\mathbb{Z}$.

**Example 10.11** Let $G$ be a split reductive group over $F$, $T \subset G$ a split maximal torus. The Weyl group $W$ acts naturally on $R(T)$ and the restriction homomorphism induces an isomorphism (see [45])

$$R(G) \cong R(T)^W = \mathbb{Z}[T^*]^W.$$  

**Example 10.12** Let $G$ be a split simply connected semisimple group over $F$. Denote by $\rho_1, \ldots, \rho_r$ the classes of the fundamental representations in $R(G)$ (with the highest weights being the fundamental ones, [13, Ch.XI]). Then $R(G)$ is the polynomial ring $\mathbb{Z}[\rho_1, \ldots, \rho_r]$ (see [1, Th.6.41], [40, Lemma 3.1], [45]).

**Example 10.13** Let $\rho_i$ be the class in $R(\text{GL}_n(F))$ of the $i^{th}$-exterior power representation of $\text{GL}_n(F)$. Then

$$R(\text{GL}_n(F)) = \mathbb{Z}[\rho_1, \ldots, \rho_{n-1}, \rho_n^{\pm 1}].$$

11. Equivariant $K$-theory

The equivariant $K$-theory was developed by Thomason in [42].

Let $G$ be a group scheme over a field $F$. A scheme $X$ over $F$ is called a $G$-scheme if an action morphism $\theta : G \times X \to X$ of the group scheme $G$ on $X$ over $F$ is given.

A $G$-module $M$ over $X$ is a coherent $\mathcal{O}_X$-module $M$ together with the isomorphism of $\mathcal{O}_G \times X$-modules

$$\rho : \theta^*(M) \cong p_2^*(M),$$

(where $p_2 : G \times X \to X$ is the projection), satisfying the cocycle condition

$$p_{23}^*(\rho) \circ (\text{id}_G \times \theta)^* (\rho) = (m \times \text{id}_X)^* (\rho),$$

where $m : G \times G \to G$ is the multiplication (see [42]).

The abelian category of $G$-modules over a $G$-scheme $X$ we denote by $\mathcal{M}(G; X)$. We set also

$$K'_n(G; X) \overset{\text{def}}{=} K_n(\mathcal{M}(G; X)).$$

If $G$ is a trivial group scheme, then $K'_n(G; X) = K'_n(X)$ are the ordinary $K$-groups of the category of coherent sheaves on $X$. 
Example 11.1 Let $E$ be a $G$-vector bundle over a $G$-scheme $X$. Then the sheaf of sections of $E$ is endowed with a natural structure of a $G$-module. Thus, we can associate to $E$ an object of $\mathcal{M}(G; X)$.

Example 11.2 Let $\mu : G \to \text{GL}(V)$ be a finite dimensional representation of an algebraic group $G$ over a field $F$. One can view $V$ as a $G$-module over $\text{Spec} F$. Clearly, we obtain an equivalence of categories $\text{Rep}(G)$ and $\mathcal{M}(G; \text{Spec} F)$. Hence there is a natural isomorphism

$$R(G) \xrightarrow{\sim} K_0'(G; \text{Spec} F).$$

Therefore, for any $G$-scheme $X$ over $F$ the inverse image map with respect to the structure morphism $X \to \text{Spec} F$ is a homomorphism $R(G) \to K_0'(G; X)$, making the latter group a module over $R(G)$.

Example 11.3 Let $H$ be a subgroup in $G$. For any representation $\rho : H \to \text{GL}(V)$ one can associate the $G$-vector bundle $P_\rho = (G \times V)/H$ over $Y = G/H$, where the $H$-action on $G \times V$ is given by $h \cdot (g, v) = (gh^{-1}, \rho(h)(v))$. We get an equivalence of categories $\text{Rep}(G)$ and $\mathcal{M}(G; Y)$. Hence, the assignment $\rho \mapsto [P_\rho]$ induces an isomorphism

$$R(H) \xrightarrow{\sim} K_0'(G; Y).$$

12. Split reductive groups

In this section we consider the case of a split reductive group. The main result is the following

**Theorem 12.1.** Let $G$ be a split reductive group defined over $F$ with $\pi_1(G)$ torsion free and $X$ be a $G$-scheme. Then there exists a spectral sequence

$$E^2_{p,q} = \text{Tor}^{R(G)}_p \left( \mathbb{Z}, K'_q(G; X) \right) \Rightarrow K'_{p+q}(X).$$

The idea of the proof is as follows. We construct the spectral sequence in three steps. First of all, we compare $K'_n(G; X)$ with $K'_n(B; X)$, where $B \subset G$ is a Borel subgroup, using the technique developed by Panin in [30]. We prove that the natural homomorphism

$$R(B) \otimes_{R(G)} K'_n(G; X) \to K'_n(B; X)$$

is an isomorphism. In the second step we prove that the restriction homomorphism $K'_n(B; X) \to K'_n(T; X)$, where $T \subset B$ is a maximal torus, is an isomorphism, using a version of the homotopy invariance theorem proved by Thomason [42]. In the last step we compare $K'_n(T; X)$ and $K'_n(X)$ by using the spectral sequence associated to a family of closed subschemes, constructed by Levine in [17]. This sequence generalizes the localization exact sequence in the equivariant algebraic $K$-theory.
Corollary 12.2.

1. The natural homomorphism

\[ \mathbb{Z} \otimes_{R(G)} K_0'(G; X) \to K_0'(X) \]

is an isomorphism, i.e. \( K_0'(X) \simeq K_0'(G; X)/J(G) \cdot K_0'(G; X) \).

2. There is an exact sequence

\[ K_2'(X) \to \text{Tor}^R(G)(\mathbb{Z}, K_0'(G; X)) \to \mathbb{Z} \otimes_{R(G)} K_1'(G; X) \to K_1'(X) \to \text{Tor}^R(G)(\mathbb{Z}, K_0'(G; X)) \to 0. \]

\[ \square \]

Corollary 12.3. Let \( H \subset G \) be a closed subgroup of a split reductive group \( G \) with \( \pi_1(G) \) torsion free. Then

\[ K_0(G/H) \simeq \mathbb{Z} \otimes_{R(G)} R(H) \simeq R(H)/J(G) \cdot R(H). \]

In particular, the group \( K_0(G/H) \) is generated by the classes of \( P_\rho = (G \times V)/H \) for all representations \( \rho : H \to \text{GL}(V) \).

\[ \square \]

Example 12.4 Let \( H \subset \text{GL}_n(F) \) be an embedding. Then for the "classifying variety" \( X = \text{GL}_n(F)/H \) of \( H \) one has

\[ K_0(X) \simeq \mathbb{Z} \otimes_{R(G)} \text{R}(\text{GL}_n(F)) \cdot R(H). \]

Remark 12.5 If \( X = G \), one has \( E^2_{p,q} = \text{Tor}^R(G)(\mathbb{Z}, K_q(G)) \). This formula together with the fact that the spectral sequence degenerates gives the computation of \( K_n(G) \) (see [17]).

Assume now that \( X \) is a smooth projective \( G \)-scheme. One can show that

\[ E^2_{p,q} = \text{Tor}^R(G)(\mathbb{Z}, K'_n(G; X)) = \begin{cases} K'_n(X), & \text{if } p = 0; \\ 0, & \text{if } p > 0, \end{cases} \]

in other words, the spectral sequence in Theorem 12.1 degenerates, i.e. \( E^2_{p,q} = 0 \) if \( p > 0 \). Hence, we get

Theorem 12.6. Let \( G \) be a split reductive group with \( \pi_1(G) \) torsion free, \( X \) be a smooth projective \( G \)-scheme. Then the natural homomorphism

\[ \mathbb{Z} \otimes_{R(G)} K'_n(G; X) \to K'_n(X) \]

is an isomorphism. Thus, the homomorphism \( \text{res} : K'_n(G; X) \to K'_n(X) \) is a surjection with the kernel \( J(G) \cdot K'_n(G; X) \).

\[ \square \]
13. Semisimple groups of inner type

Let $G$ be a simply connected semisimple group over $F$ and $\rho : G \to \text{GL}_1(A)$, be an algebraic group homomorphism, where $A$ is a central simple $F$-algebra. Since $A_{\text{sep}}$ is a matrix algebra over $F_{\text{sep}}$, $\rho_{\text{sep}}$ is a representation of the group $G_{\text{sep}}$. We call $A$ a Tits algebra of the group $G$, if $\rho_{\text{sep}}$ is an irreducible representation over $F_{\text{sep}}$. A Tits algebra $A$ is uniquely determined, up to an isomorphism, by the representation $\rho_{\text{sep}}$ (see [45]).

Denote by $C$ the center of $G$. It is a group scheme of multiplicative type. The restriction of the homomorphism $\rho$ to $C$ is given by the multiplication by some character $\chi \in C^* = \text{Hom}(C, \mathbb{G}_m, F)$. Moreover, any character in $C^*$ can be obtained in such a way. The class of the corresponding Tits algebra $A$ in the Brauer group $\text{Br}(F)$ does not depend on the choice of $\rho$, but only on the restriction of $\rho$ to the center $C$. Thus, we get the Tits homomorphism

$$\beta : C^* \to \text{Br}(F).$$

A Tits algebra $A$ is called a fundamental Tits algebra if $\rho_{\text{sep}}$ is a fundamental representation. If $G$ is a group of inner type, then the fundamental Tits algebra exists for any fundamental representation of $G_{\text{sep}}$.

Example 13.1 The fundamental Tits algebras of $\text{SL}_1(A)$, where $A$ is a central simple $F$-algebra of degree $n$, are the $\lambda$-powers $\chi^i A$ of $A$, $i = 1, 2, \ldots, n - 1$ (see [15]).

Let $\rho : G \to \text{GL}_1(A)$ be a group homomorphism, where $A$ is a central simple $F$-algebra. We denote by $\rho A$ the algebra $A$ together with the $G$-action given by the formula

$$ga = \rho(g) \cdot a \cdot \rho(g)^{-1},$$

where $a \in A$ and $g$ is in $G$. We simply write $A$ for the algebra $A$ together with the trivial $G$-action.

Let $X$ be a $G$-scheme. We consider the abelian category $\mathcal{M}(G; X, A)$ of $G$-$A$-modules and morphisms of $A \otimes_F \mathcal{O}_X$- and $G$-modules and set

$$K'_n(G; X, A) = K_n\left(\mathcal{M}(G; X, A)\right).$$

The exact functor (being an equivalence of categories)

$$\mathcal{M}(G; X, A) \to \mathcal{M}(G; X, \rho A),$$

taking $M \in \mathcal{M}(G; X, A)$ to the same $A \otimes \mathcal{O}_X$-module $M$ with the $G$-action given by

$$g \cdot m = \rho(g) \cdot gm,$$

induces an isomorphism

$$K'_n(G; X, A) \cong K'_n(G; X, \rho A).$$

The image of an element $u \in K'_n(G; X, A)$ in $K'_n(G; X, \rho A)$ we denote by $\rho u$. 
Denote by \( \langle u \rangle \) the image of an element \( u \in K'_n(G; X, \rho A) \) under the homomorphism

\[
\text{for } : K'_n(G; X, \rho A) \to K'_n(G; X)
\]

induced by the forgetful functor

\[
\mathcal{M}(G; X, \rho A) \to \mathcal{M}(G; X).
\]

**Theorem 13.2.** Let \( G \) be a simply connected semisimple group of inner type, \( X \) be a \( G \)-scheme. Then the restriction homomorphism \( K'_0(G; X) \to K'_0(X) \) is surjective, and its kernel is generated by the elements \( \langle \rho u \rangle - \langle u \rangle \) for all fundamental Tits representations \( \rho : G \to \text{GL}_1(A) \) and all \( u \in K'_0(G; X, A) \).

As an application we compute the Grothendieck group of an adjoint group of inner type.

**Theorem 13.3.** Let \( G \) be an adjoint semisimple group of inner type over \( F \), \( \pi : \tilde{G} \to G \) the universal covering, \( C = \ker(\pi) \). Then the natural homomorphism

\[
\frac{R(C)}{J(\tilde{G}) \cdot R(C)} = \mathbb{Z} \otimes_{R(\tilde{G})} R(C) \to K_0(G)
\]

is an isomorphism.

Consider the isomorphism in the Theorem in detail. First of all, note that \( R(C) \cong \mathbb{Z}[C^*] \). For any character \( \chi \in C^* \) consider the line bundle \( P_\chi = (\tilde{G} \times A_{\chi})/C \) on \( G \), where the \( C \)-action is given by \( c \cdot (\tilde{g}, x) = (\tilde{g}c^{-1}, \chi(c)x) \) and denote by \( n_\chi \) the greatest common divisor of dimensions of all representations of \( \tilde{G} \) over \( F \) with the restriction on \( C \) given by the multiplication by \( \chi \). These numbers depend only on the Dynkin diagram of \( G \) and were computed in [26].

**Corollary 13.4.** The assignment \( \chi \mapsto [P_\chi] \) induces an isomorphism

\[
\mathbb{Z}[C^*] / A \cong K_0(G),
\]

where \( A \) is the subgroup in \( \mathbb{Z}[C^*] \) generated by

\[
n_\chi \psi^{-1} \cdot \text{ind}(\chi \psi^{-1}) \cdot (\chi - \psi)
\]

for all \( \chi, \psi \in C^* \) (here \( \beta : C^* \to \text{Br}(F) \) is the Tits map). In particular, the group \( K_0(G) \) is generated by the classes of linear vector bundles \( P_\chi, \chi \in C^* \).

**Example 13.5** Let \((V, q)\) be a quadratic form over \( F \) of dimension \( 2n + 1 \), \( G = \text{O}_+(V, q) \) be the special orthogonal group. Then \( \tilde{G} = \text{Spin}(V, q) \) is the universal covering group, \( C = \mu_2, C^* = \{0, \chi\} \) and \( n_\chi = 2^n \) (see [26]). The fundamental Tits algebra corresponding to the character \( \chi \) is the even Clifford algebra \( C_0(V, q) \). Hence

\[
K_0(\text{O}_+(V, q)) = \mathbb{Z} \cdot 1 \oplus \left( \mathbb{Z}/2^n \text{ind} C_0(V, q) \mathbb{Z} \right) \cdot ([P_\chi] - 1).
\]
14. Arbitrary reductive groups

Let $G$ be an arbitrary reductive group over a field $F$. We would like to find a condition on $G$ under which the restriction homomorphism

$$K'_0(G; X) \to K'_0(X)$$

is surjective for all $G$-schemes $X$. Assume that taking $X = G_E$ for some finite field extension $E/F$, we get a surjection, i.e. the homomorphism

$$Z = K_0(E) = K'_0(G; G_E) \to K'_0(G_E)$$

is surjective. Therefore, $K'_0(G_E) = Z \cdot 1$. Hence, the first term of the topological filtration $K'_0(G_E)^{(1)}$ of $K'_0(G_E)$ (see [36, §7.5]), being the kernel of the rank homomorphism $K'_0(G_E) \to Z$, is trivial, in particular,

$$\text{Pic}(G_E) \simeq K'_0(G_E)^{(1/2)} = 0,$$

i.e. $G$ is a factorial group.

Surprisingly, the converse is true.

**Theorem 14.1.** Let $G$ be a reductive group defined over a field $F$. Then the following conditions are equivalent:

1. $G$ is factorial.
2. For any quasi-projective $G$-scheme $X$ the restriction homomorphism

$$K'_0(G; X) \to K'_0(X)$$

is surjective.

**Corollary 14.2.** Let $H$ be a subgroup in $G$. Then $K_0(G/H)$ is generated by the classes of vector bundles $(G \times V)/H$ over $G/H$ for all linear representations $H \to \text{GL}(V)$.

**Remark 14.3** One can drop the assumption for $X$ to be quasi-projective in the case when the factor group $G'/G''$ modulo the commutant is a quasi-trivial torus. For example, a semisimple group satisfies this property.

At the end of this section we consider the smooth projective case:

**Theorem 14.4.** In the conditions of Theorem 14.1 let $X$ be a smooth projective $G$-scheme over $F$. Then the restriction homomorphism

$$K'_0(G; X) \to K'_0(X)$$

is split surjective.

15. Applications

In this section we consider some applications of the mentioned results.

**Theorem 15.1.** Let $G$ be a reductive group over $F$, $H$ be a connected subgroup in $G$, $X = G/H$. Then the group $K_0(X)$ is finitely generated.
COROLLARY 15.2. Let $G$ be a reductive group over $F$. Then the kernel $K_0(G)^{(1)}$ of the rank homomorphism $K_0(G) \rightarrow \mathbb{Z}$ is a finite group. In particular, $K_0(G)$ is a finitely generated group of $\mathbb{Q}$-rank 1.

Remark 15.3 The group $K_0(G)$ for an algebraic torus $G$ was computed in terms of generators and relations in [27].

Let $G$ be an algebraic group and $X$ be a $G$-scheme over $F$. For any $g \in G(F)$ denote the automorphism $x \mapsto g \cdot x$ of $X$ by $\lambda_g$. The group $G(F)$ acts naturally on $K'_n(X)$ by the inverse images $\lambda_g^*$.  

PROPOSITION 15.4. Let $G$ be a reductive group. Then

1. For any quasi-projective $G$-scheme $X$ the group $G(F)$ acts trivially on $K'_0(X)$.

2. For any smooth projective $G$-scheme $X$ the group $G(F)$ acts trivially on $K'_n(X)$ for any $n \geq 0$.

COROLLARY 15.5. The classes of rational points from the same $G(F)$-orbit in $X(F)$ coincide in $K'_0(X)$. In particular, the classes of rational points of $G$ in $K_0(G)$ are pairwise equal.

References


