Mathematics is one of the oldest scientific fields and it is certainly the most widely taught subject in schools all over the world. As mathematicians we have great responsibilities towards the education in mathematics of future generations. What we teach in the classrooms or present in lectures, is not only important for the immediate mathematical needs of the students but has, in the longer perspective, a decisive impact on the image of mathematics in society. Actually mathematics meets a real need of modern societies and individuals.

It is not enough that researchers in mathematical education or didactics of mathematics think about these matters. It is important that all teachers of mathematics – also at the university level – think about them and, above all, are ready to take initiatives to improve the teaching of mathematics at all levels, where necessary. The purpose of the Round Table on Education was to stimulate discussion and awareness among mathematicians of the fundamental importance of the teaching of mathematics.

Issues for the Round Table on Education

1. The contribution of the didactics of mathematics to our understanding of mathematics as a science and a subject.
2. Basic issues of mathematical education.
3. Mathematics in primary and secondary schools, and recruitment to university.
5. European summer schools for secondary school teachers.
6. Current problems in the teaching of mathematics at the university level.
7. Computerized tests for mathematics qualifications.
8. What are the challenges for the teaching of mathematics in schools and at universities in the 21st century?

Structure of the Round Table on Education

The oral contributions to the round table included two invited presentations on the above issues and questions, and a few other invited contributions in order to open the discussion on issues not covered by the presentations. Following each invited contribution, participants were free to respond to the issues raised. After the round table some of the participants submitted further comments, which have
been included in this report together with the written accounts of the invited contributions and extracts from the discussions.

The two invited presentations were:

**Lajos Posa:** *Convictions and doubts concerning mathematical education.*

**Jean-Pierre Boudine and Christian Mauduit:** *A European Knowledge Certification Network System.*

The other invited contributions were:

**Vagn Lundsgaard Hansen:** *Mathematics in primary and secondary schools, and recruitment to university.*

**Miklos Laczkovich:** *Current problems in the teaching of mathematics at the university level.*

**Vagn Lundsgaard Hansen:** *New challenges for the teaching of mathematics.*

To supplement the report on points not sufficiently covered by the invited contributions and discussions, a few persons were subsequently invited to present their views on some of the issues for the round table. These contributions are placed at the end of the report and include the following articles:

**Vinicio Villani:** *Feedback from mathematics education to mathematical research.*

**Mogens Niss:** *The contribution of the didactics of mathematics to our understanding of mathematics as a science and a subject.*

**Jean-Pierre Kahane:** *Mathematics education towards mathematics.*

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**Convictions and doubts concerning mathematical education**

**Lajos Posa**

Eötvös University, Budapest (Hungary)

The world over, mathematics is taught as an *important* subject. Why is it so important? I would like to believe (as many of us do) that mathematics is important because it trains children to think. Indeed, much sooner than other branches of learning, and requiring very little knowledge, mathematics provides the possibility of creative thinking at a high level, unrestricted by authoritarian principles, using common sense and the flight of the imagination.

But have these hopes been realized? I am afraid that the freedom and pleasure of thinking have not been imparted to children. This has remained an empty slogan, a mere declaration of our importance. Meanwhile, in everyday life, and in some schools, mathematics is of a low standard, boring and empty; elsewhere it is too difficult and remains the source of constant frustration. Mathematics is seldom what it should be. Why is this so? Here are some possible causes:
1. We have not gone beyond the traditional schooling based on the exposition given by the teacher. This old-fashioned attitude emanates from textbooks as well as from the teachers themselves. No wonder: they were taught in the same way.

2. The usual subject matter of mathematics is not suited to the purpose of original thinking and learning through exploration and discovery. I am afraid that the ‘usual’ fundamentals that ‘should be’ taught according to adult society are hopelessly difficult for a significant portion of children. To take just one example: for many children aged twelve, computations with complicated fractions are nothing but phantasms lacking any real meaning. Perhaps some unusual topics – like games of strategy – would be more suitable for our purposes?

3. Is mathematics really for everyone? Is the chance of joyful and successful thinking really granted to every child? Or should mathematics be the amusement of children fascinated by this topic, as is the case in music, drawing and sports? (For today’s school children it is much more frustrating not being good at mathematics than, say, being unable to sing at the same standard.)

And, finally, another fundamental question. Supposing that a child is a good and effective mathematical thinker, does this really improve their thinking in some deeper sense of the word? Does this faculty transcend itself? For example, does it transfer to a better understanding of other subjects? Does it really have benefits of any kind?

I would like to believe in this transfer, and also that it is practicable for every child. Or, at least, that it will be practicable, if we do it better.

I have also been asked to say a few words about my work with mathematics for talented children.

Unlike the mass of children frustrated by failure in mathematics, there are children who easily learn the mathematical curriculum of normal schools. They are bored by the curriculum and they could easily discover much more. It is those children that I here, for short, call “talented”. There are many such children, according to my personal experience.

The problem of the mathematical education of talented children is unsolved. As for special classes, I think on the one hand that we need them, but on the other hand that they do not solve the problem, since: (i) not every talented child is admitted to such a class; (ii) this form of education can be harmful for some children (when they are not among the best in such a class).

It is frequently asked why one should care about the mathematical development of young children aged 12–13. After all, we do not need that many mathematicians! The answer is that for children aged 12–13, talent, as defined above, usually is NOT exclusively mathematical. However, at this age the well-developed intellect can manifest itself only in mathematics; it is mainly in this field that the mind can produce arguments at a high level. When we teach these children, we therefore educate the intelligentsia (or, more generally, thinking people) of the future. This makes the responsibilities in connection with this work so great.
From the discussion

Helge Elbrønd Jensen, Denmark: It is well known that Hungary, like other East European countries, has been successful in training talented children to obtain good and even extraordinarily good results in international competitions in mathematics. But now I ask: What actually happens to these children afterwards? Do they become creative mathematicians either as researchers or teachers? Is it from these special classes that society recruits the persons who take the leadership in research and teaching as adults?

Lajos Posa, Hungary: Only very few children from the special classes go on to study mathematics. Typically, they study engineering or economics or something like that. One should not advise children to come to the special classes unless they are really good at mathematics, say among the top 10% in their class.

Per Hag, Norway: It should be emphasised that today, more than ever, mathematics (and statistics) is an important, and in many cases indispensable, tool in different fields: engineering, science, economics, ecology, medicine and biology. As is stated in the well-known Davis report [Notices Amer. Math. Soc., October 1984] on the situation for mathematics in the US: “Too few people recognize that the high technology, so celebrated today, is essentially a mathematical technology.”

To say that mathematics is important because it trains children to think is, therefore, misleading. One should rather stress the fact that teaching mathematics (and statistics) is important, foremost, because mathematics is the basic tool in many professions today. Collaboration between teachers and experts in the various fields of applications of mathematics is necessary in this connection!

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A European Knowledge Certification Network System

JEAN-PIERRE BOUDINE AND CHRISTIAN MAUDUIT

Université de la Méditerranée, Marseille (France)

The system we want to present is a response to an initiative of the European Community Commission. On November 26, 1995, the commission published a book entitled Teaching and learning, toward the Cognitive Society, signed by Edith Cresson, Padraig Flynn and Martin Bangemann. One of the proposed goals described in this book is “the support of the acquisition of new skills”, and related to this goal, “to create new kinds of skills certification”. Professor Jean Louis Reiffers, Université de la Méditerranée, France, who was one of the editors of the White Book providing background material for the initiative of the ECC, asked us to work on the concrete realization of this project in the case of mathematical knowledge. Together with a few other French and Italian colleagues, we therefore began to work out computerized tests in mathematics, respecting the highest possible safety regulations and living up to recognized evaluation criterions. The tests are based on automatization in order to make them easily accessible, to secure objectivity, and to keep the costs of the whole system low.
Anybody should be able to take those tests at almost any time and at a designated place within easy reach. The use of a computer should guarantee equal chances to all candidates, since there will be no more or less demanding, well-meaning, or exhausted examiners.

**The target group**
The target group for this testing system is really the whole population of the European Community, with no age limit. Access will be granted to everyone who wants his knowledge in a particular field to be tested and certified. The results obtained will be stored on a magnetic card. The recognition of the results of these tests by institutions and companies will depend on the existence of a consensus on the objective value of the procedure. In a sense, the system can be thought of as a broad generalization of tests like the TOEFL.

In our opinion, such a system will not represent any threat to recognized academic diplomas. In fact, the system of knowledge certification and academic diplomas should not be considered as rival systems, but as complementary systems. Moreover, it could be of some help on the issue of foreign diploma equivalence, but this is really a minor point in relation to the main purpose.

It is generally believed that the users of computerized testing systems will most often be people who need to acquire and validate new competences after their entry into professional life. It seems reasonable to believe that making the access to certification easier will strongly motivate people to undertake training, and thus increase the whole level of knowledge in Europe.

**Certification centers**
Once the system has been established (in mathematics as well as in other fields), a network of companies will be in charge of its management. The first task of this consortium will be to advertise the system in every European country with the authorities who will arrange the certification centers: Universities, Chambers of Commerce, city councils and other local administrative boards. We plan several thousands of certification centers, and several millions of examinations per year.

**Description of the project for mathematics**
The level of the tests must range from the most elementary level (adults with no diplomas or no qualifications) to a qualified technician level (that is to say the first two years of University).

1. One has to structure the field of mathematical knowledge into a limited number of *units*. At the moment we have about thirty units in mind, encompassing such things as: Elementary operations with numbers, ordering, ratios, elementary logic, polygons and basic geometry, basic trigonometry, functions and polynomials, calculus, finite probabilities, Fourier analysis, etc.

This phase is of tremendous importance and difficulty, since we do not want the construction to be relevant to one particular educational system only, BUT for the employment market as a whole, i.e. for use in society. It is clear that we should also recognize mathematical skills other than those specifically, or implicitly, tested or rewarded in schools. For example, the ability to work with a technical drawing, to perform data analysis, or to use computer software for symbolic computations.
2. The second phase is the elaboration of the set of knowledge units. They will include open questions (with numerical or algebraic answers) as well as multiple choice questions (with a non-constant number of answers, including sometimes several exact answers). We will also take into account whether the answers (even false) are consistent with each other throughout the whole test. Moreover, it will be necessary to create sufficiently many variants, for each kind of question in each test unit, and to renew this set of questions very often. This will be important, in order to avoid the possibility of cheating, and to make sure that candidates will not just be trained to answer some stereotyped form of questions, but will obtain a real knowledge of the field tested by the unit.

3. The certification process will be the same for mathematics and every other field involved in the whole program. The certification process could be as follows. The candidate, after identification by a local authority, will sit in front of a terminal, insert his card and ask for a particular unit. The computer will be connected to a remote server by the internet and will receive the questions in an interactive form. At the end of the exam, the form will be sent back to the server. The server will mark the answers and send back the results, updating the record card.

The prototype
Up to now, we have started working on those three parts in order to build a prototype. Angelo Lissoni and the Italian team has developed the third part (the computer system). In collaboration with several colleagues, we have sketched a set of thirty units and gathered a few test collections. However, our work should be considered only as a first draft. It is clear that defining the content of each unit in the first place and then test-editing it, is a huge task that requires the participation of specialists from many countries already involved in this kind of work.

The Leonardo Project
Obviously, we need to have a larger framework for this initiative. In particular, we need to involve more people in it. This is the reason that we have applied for support from the European Community through a Leonardo Project. The promoter of the project will be Université de la Méditerranée (Aix Marseille 2), which has accepted to be in charge of it. The international consortium counts among its members:

- the University of Cambridge Local Examinations Syndicate (United Kingdom),
- the Institut für Techno- und Wirtschaftsmathematik e.V. from Kaiserslautern (Germany),
- the Department of Computer Science of Trinity College from the University of Dublin (Ireland),
- the Department of Mathematics and Computer Science from Uniwersytet Mikolaja Kopernika from Torun (Poland),
- the Centro Eleusi of the Universita Commerciale Luigi Bocconi from Milan (Italy),
several publishers, among which Springer Verlag from Heidelberg (Germany), but also Quadrature (France) and Sigma Editura (Rumania),
several associations, among which QCM Pro from Ecole Supérieure d’Ingénieurs en Electronique et Electrotechnique and Math pour Tous.

From the discussion

K. Császár, Hungary: Examinations of students by computerized tests can be used only to identify those who are not able to reach even the lowest level of knowledge and so have to fail. By a simple test – after my experience – you can check only whether the student is able to apply a theorem, whether he has clearly understood its content, but it is impossible in a simple way to evaluate logical thinking, i.e. the ability to go through a chain of thoughts. A test may be good as an entry for a written exam but has to be followed by an oral exam where you can check the intellectual power of students who would not become specialists in mathematics but creative specialists in some other branch of science and technology.

Exams organized in this way need a bigger personnel teaching mathematics, you have to work much more at the exams, but it is worth the trouble.

Azad Tagi-zade, Azerbaijan: Experience gained from mathematical olympiads indicates that there is a good correlation between ‘test’ results and traditional methods for assessment of mathematical knowledge.

Gerhard König, Germany: Computerized tests can be used for testing knowledge, but as everybody knows, there is a probability that the correct answer is given by guesswork. This can, of course, be taken care of by demanding \( x \) answers right from \( n \) distractors. How do we know, however, that the person being tested understands the problem, or whether he or she only marks an answer? What information will truly indicate how the person being tested has done the problem? Are only those test items chosen which can be easily formulated, but are not important for the subject? Have test experts (from psychology) been engaged in the formulation of the problems, in the actual testing process and in the evaluation of results?

Miklos Simonovits, Hungary: I think that computers may have an important role in many parts of our educational system, e.g. in teaching data connected with history and in the teaching of foreign languages.

I think that teaching mathematics is very complicated and that computers may be useful in developing notions in calculus, geometry, etc. But I do not believe that computers are good in checking knowledge to replace oral exams and traditional written exams.

A Hungarian student in our educational system has often to write half a page to answer one single question in a test. Often we give 4 – 5 questions for a 1 hour test. I do not believe that in the near future computers will be good at checking these answers. Allowing computers in the examination procedures will therefore force us to change the whole educational system from better to worse.

The question of testing unemployed people is a marginal problem in this context.
John Erdos, UK: One must always keep in mind that the tests and the mode of testing always influence teaching. There is a danger that teaching and especially learning, will be aimed at the material and the skills required by the tests. The more successful and widely used the testing system is, the greater the danger.

The argument that the results of some new test correlate well with the results of traditional tests is totally specious; it is virtually self-evident that it will be so, since the rank order of students on any reasonable test will reflect their talent and education. But if future students aim at tests that only verify that they have learnt simple skills, then the good education that is required to do well in traditional tests might go out of the window.

I do feel that there is a place for computer-based testing and uniform testing over a large population. Apart from the obvious cost advantages, it would be useful for things like comparison of standards. With other tests, such comparisons are more difficult. However, to avoid the dangers I outlined, computer tests – or other types of simplified or “objective” tests – should be combined with other forms of assessment. These forms should, as far as possible, be direct tests of the skills and abilities we value and not some quantity that we hope might correlate with them.

Miklos Laczkovich, Hungary: If we enlarge the discussion on computerized tests to the broader subject of computer-assisted education (and programs teaching calculus, etc.), I should like to remark that (notwithstanding the obvious usefulness and importance of these programs), my experience is that at the very first and very basic level of university mathematics education, there are problems where they cannot really help. What I mean is that the basics of university level mathematics education are: understanding the logical operations (and, or, not, if, if and only if), understanding exactly what we mean by a mathematical statement, by an implication, by a proof, and by mathematics in general. This is very difficult for most students, since they have seen nothing similar at secondary school. To make them understand these basic notions and methods, they need many examples, problems, personal attention, and I doubt if computer programs can be of much help in these basic questions.

Boudine and Mauduit, France: We shall try to give answers to some of the critical remarks and comments put forward in the discussion.

First of all, it is not very surprising for such a new subject, that certain misunderstandings tend to put off the true centre of the discussion. The system of tests by computer which we are building ought to provide validation, not for basic university education, but for professional training. Nor should it provide validation for basic school education, but for further training, not only in mathematical fields, but also in mathematical skills and knowledge fundamental for professionals. In fact, it concerns validation of knowledge of anybody who has to change region, country or profession.

Our colleagues are right in thinking that this system of tests by computer has some inherent limitations. By definition, the system is not allowed to know the capacity of a person for oral and written expression. While this limitation is evident, the other limitations are less obvious and the best present-day tests avoid the ones which have been mentioned. For example, we can in practice easily
prevent the strategy of random answers and we can also know if correct reasoning was applied. Moreover, it is probable – in fact, it is most often the case – that before a vacancy is filled, the employer will complete the validation of professional qualifications by an interview and/or a written examination.

By way of conclusion, we want to emphasize the deep social significance of this project, which could be of prime importance for the development of competencies in Europe.

As a follow up to the discussion, Boudiné and Mauduit invited mathematicians to join a group with the purpose of working out sufficiently varied and suitable testing problems in mathematics for the computerized system.

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Mathematics in primary and secondary schools, and recruitment to university

Vagn Lundsgaard Hansen

Technical University of Denmark, Lyngby (Denmark)

In the last decade many countries have experienced a decrease in the number of students enrolling for university studies in the natural and engineering sciences, with biology as the only exception. Many explanations have been given. Some people think that teaching in primary and secondary schools does not stimulate interest in these subjects and, furthermore, that it does not prepare students properly for university studies. Other people think that changes in society and attitudes to scholarship, caused among other things by the influence of television, make it difficult to motivate students to study subjects which need hard and dedicated work without immediate rewards except for intellectual insight. A third factor, which is regularly mentioned as a reason, is that it is no longer prestigious to study these subjects; the ‘heroes’ in society are not scientists but people in business and directors: money before intellect.

There is probably not one single cause of the problem. And as a matter of fact, what is the problem? There are more students than before studying these subjects, except for a short period in the 1960s and 1970s. The point is rather that the total number of students going to university has grown enormously and that the growth has primarily taken place in other subjects. Since most governments link the budgets to the number of students at the various faculties, the natural and engineering sciences certainly have a financial problem. Aside from this financial problem, the real question for me therefore is not the relative decrease in the number of enrollments but rather whether we are getting the bright and motivated students. The absurd thing is that the subjects where mathematics and physics are really fundamental are often the subjects where the entrance requirements are lowest.

The above considerations are valid in my own country, Denmark. I suspect they are valid more generally. I hope the discussion will clarify this. In any case, I would find it useful if we could get short reports from various European countries.

My own experience is that it is very difficult to have a discussion on the role of teaching mathematics in primary and secondary school in relation to the
recruitment of students to university studies in which mathematics plays an essential role. The tendency is that the discussion rather quickly ends up with mutual accusations between the various levels of the educational system. It would indeed be helpful if mechanisms could be found for having a balanced discussion on this vital point.

Comments and short reports from various countries

Miklos Laczkovich, Hungary: Sometimes the different levels of mathematical education blame each other: universities and colleges claim that their education is not effective because the level of high school students is low, so they have to deal with masses of mediocre students. On the other hand, high schools complain that the level of mathematics teacher education is low, so they have no good teachers. This vicious circle should be stopped by increasing the level of education at universities and colleges. The education of good teachers should come first, and governments should realize that high priority must be given to good university and college education.

Christian Mauduit, France: I want to mention some activities aimed at a large public that have been developed during the last years, mainly in France, but not only in France. For example, the Kangaroo competitions, based on an idea coming from Australia and Canada, since 1991 have annually involved from 100,000 up to 700,000 children in France, and with several thousands of participants in many other European countries. The most important characteristic of these kinds of activities is that they are not exclusively devoted to the best students, or to the most motivated children. Rather, the goal is to show that mathematics can be attractive to everybody, even to students unable to get good results at school.

One of the most interesting projects in this direction has been developed by Pierre Audin and Pierre Duchet since 1990. A description of the project, called MATh.en.JEANS, can easily be found in the many publications of Audin and Duchet. Let me here just say that its goal is to give young people the possibility of working together with teachers from high schools and professional mathematicians on open questions in mathematics.

Even though I strongly believe in the importance of the development of mathematical activities for everybody, I want to call the attention of the mathematical community to the risk of a sharp drift of these programs. As soon as such an activity attracts significantly many people, a lot of money can be involved (e.g. in the case of the Kangaroo competitions, between 1 and 2 million USD each year). If we are not careful, this situation can attract people or commercial enterprises which, on the pretext of developing popularization, could have as their first goal the making of huge profits (sometimes from the work of benevolent or underpaid colleagues). It is time now for us to decide what rules must govern the connections between money and popularization in mathematics, and I strongly hope that the EMS will soon open a discussion on this very important problem.

Klaus D. Bierstedt, Germany: Two ‘caveats’ at the beginning: Some of my statements below tend to be exaggerated, due to lack of space. Also, in Germany education and universities are a matter for the Federal States, not for the Federal
Republic; thus, the situation has always been different at universities in different states of the Federal Republic.

Quite generally, however, in recent years the number of students who would like to get a diploma degree in mathematics seems to be too small. More students want to obtain the so-called ‘(Erstes) Staatsexamen (für die Sekundarstufe II)’, to become a teacher of mathematics and some other topic in a Gymnasium, but the amount of mathematics they learn is much smaller, a tendency which has become stronger and stronger during the last decades.

Let me explain this point in more detail, concentrating on the situation in the biggest German state (Nordrhein-Westfalen) and on the curriculum at my own university (Paderborn), which is trying to make the best out of the existing state law. Students who would like to get the ‘Staatsexamen’ take the courses ‘Analysis I, II’ and ‘Linear Algebra I, II’ during the first two semesters. Then they have one course in applied mathematics (i.e., numerical analysis or stochastics or differential equations) and a shorter course on ‘mathematics with a computer’, before passing the intermediate exam (‘Zwischenprüfung’). In the rest of their time at the university (in the so-called ‘Hauptstudium’), they take one course on each of the following topics: applied mathematics, complex function theory or functional analysis, topology or geometry, algebra or computer algebra or number theory, and they have to pass one seminar in mathematics. In didactics of mathematics, they would also have a shorter course (didactics of analysis or linear algebra or stochastics) and a seminar.

It is quite clear that these students do not know very much mathematics when they become teachers. They do not get a good picture of what is happening in modern mathematics, or of its many applications. E.g., since they do not usually take the second semester of a course in, say, functional analysis, they will not see the deeper theory, nor most of the more interesting applications. But this is only part of the problem: The students for a ‘Staatsexamen’ take courses from three different departments; they do not even think of themselves as being ‘math students’. They try to judge each course in mathematics mainly by its possible value for what (they believe) they might have to teach in school later on. Thus, for both students and professors, the situation is all but desperate, and there does not seem to be a simple ‘remedy’.

Luc Lemaire, Belgium: In the 1980s, Belgian universities experienced a sharp decline in the enrollment of students in mathematics, both in total number and in the number of very bright students. Surprisingly, it was possible to reverse that trend in almost a single year (1990) by various information campaigns.

Secondary schools in Belgium have undergone major financial cuts, and the situation of teachers has deteriorated. Not many young people want to become teachers, and those who do, fear they will not get jobs. In fact, most secondary school students used to think that studies in mathematics only led to a career in teaching, and this was a major reason for the weak recruitment.

Almost none of the very bright students knew that mathematics was a living science, and hence they turned to physics or biology excited by the idea of a research career. The action we have taken has been to find out the careers of (almost) all former students over the last 10 years. At the time (1989) all had
jobs, evenly spread between banking, teaching and research. When this was made
known in a very precise article in a major newspaper, the effect was immediate.
At my university we moved from 35 students in the first year to 70 and this is
an essential progress for us. We have also been successful in getting into contact
with the very bright students at the finals of the olympiads, for example, giving
informal lectures on contemporary mathematical research.

My advice for improving recruitment is therefore: 1) Provide information on
the variety of careers mathematics can lead to, including the actual job situation;
2) Be as precise as possible, do not stick to general statements, and always tell
the truth about the job market (the future of young people should come first); 3)
Explain whenever possible to bright students that mathematical research exists,
by talking about some present day problems.

I think this kind of information campaign must be done for mathematics by
itself, and not for scientific studies in general. Indeed, it is an important fact that
mathematics is used in insurance and banking, whereas other sciences are not.
Hence it does not make sense to use the same arguments for all sciences.

Concerning the level of secondary school teaching, the financial cuts in Bel­
gium are so drastic that many teachers will get sacked in 1996. Against all odds,
highly motivated teachers fight desperately to maintain standards. I suspect (hope)
that the situation is worse in Belgium than in the rest of Europe.

Kari Hag, Norway: Only 13% of the full age group takes the highest level math­
ematics in Norwegian secondary school. This percentage has remained relatively
constant over the ten year period 1986–96. The change in demography in Norway
means, however, that the number of 18 year olds has dropped almost 25% over the
same period. Moreover, as in other countries, the percentage of students studying
physics decreases, while the number of those studying biological sciences increases,
despite the drop in the total number of secondary school students.

Though recruitment in science and engineering at the Norwegian universities
has remained good, colleges are struggling to fill their quotas and at the engineering
colleges one out of four places is vacant.

In Norway students do not enter a university or college to study mathematics
(or any other subject) as an undergraduate major. In comparison with the situa­
tion in other countries, very few students go on to graduate studies in mathematics,
which means entering a Master’s Program. Popular areas of study at this level are
applied and industrial mathematics including statistics followed by pure mathe­
matics. Unfortunately, practically none of the Master’s Degree students want to
become teachers. Indeed, of the last 70 candidates at the University of Oslo only
2 are now secondary school teachers.

Although relatively few students finish the Master’s Program, which is a
requirement to enter a PhD Program, the number of students who receive a PhD
in Mathematics is comparable to those in other countries, for example, the United
States.

How can we increase the number of students who specialize in mathematics?
More information for possible careers is needed. For example, the results of the
Belgian information campaign are encouraging. In addition, a long term project is
to raise the quality of the mathematics teachers in primary as well as secondary
schools. As in most other countries, the government is introducing reforms at all school levels. However, no reform will succeed unless good teachers are available.

David Salinger, UK: In England and Wales, the number of undergraduate students studying mathematics has increased by about a third since 1985–86, the first year for which the data is available. Though the last figures I have are for 1993–94, I am fairly confident that there are now about ten thousand UK students studying mathematics as a first degree in the “old” universities. Over the same period, general recruitment to universities has risen more quickly, particularly among women. If it can be done, we should encourage more women to continue into university mathematics: they form a third of mathematics students, but more than half of university students as a whole.

As far as teacher education is concerned, the picture is one of chronic under-recruitment. For example, fewer than half the mathematics teachers in secondary schools in England have a university degree with mathematics as a main subject.

If the figures for the number of students entering university to study mathematics are mildly encouraging, it is another matter when we consider the mathematical level of starting students. A recent report [Tackling the Mathematics Problem, London Mathematical Society, October 1995] stated that mathematicians, scientists and engineers in Higher Education perceived that those students had: (i) a serious lack of essential technical facility – the ability to undertake numerical and algebraic calculation with fluency and accuracy; (ii) a marked decline in analytical powers when faced with problems requiring more than one step; (iii) a changed perception of what mathematics is – in particular of the essential place within it of precision and proof. For a passionately-argued view, the interested reader may consult Wrong way. Go back! and Back to the future and “Problem solving” or Problem Solving? by A. Gardiner in [Mathematical Gazette, Nos. 485–487, 1995–1996].

Surprisingly similar concerns are shared by Marc Rogalski [Le nouveau public étudiant en sciences: quels choix stratégiques? Gazette des Mathématiciens, 69, Juillet 1996, 39–44] and others in the French mathematical community. Moreover, this round table debate shows that the issue is not confined to a couple of European countries.

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Current problems in the teaching of mathematics at the university level

Miklos Laczkovich

Eötvös University, Budapest (Hungary)

I shall restrict myself to talk about the undergraduate level only. At this level, the general problems in teaching are mainly related to two circumstances: (i) the lack of time; (ii) the sizes of groups.

As for (i), the pressure of limited time is always in conflict with the desire to cover as much material as possible. Some teachers express the opinion that every university course should contain some new results of the subject, and should give a
review of the modern developments. Unfortunately, sometimes we have to give up these hopes. Usually we have to restrict ourselves to the basics. Besides, a deeper knowledge of the basics is much more important than a superficial glance at some sophisticated results which will be forgotten anyway.

With respect to (ii) the sizes of classes should be as small as possible. This is of course a purely financial problem, but it can be crucial, if we want to maintain a certain level of the education.

*Remarks on the teaching of mathematics at the university level*

**K. Császár, Hungary:** It is certain that mathematicians at the Technical University Budapest, as well as at Eötvös University in Budapest, would like to teach a few branches of mathematics in depth with greater possibility to continue developing the students’ creativity, and the ability to apply knowledge learned to the solution of new problems. I think, wherever mathematics is taught as a background subject, and not as a subject in its own right, our task is first of all to teach basic mathematics needed by the experts teaching branches applying mathematics. If we leave it to these experts in other fields to teach the basic mathematics, students will often not be able to apply a mathematical theorem or method to any other problem than that in which he or she first met it. Talking briefly about important theorems without proving them is hard work for mathematicians as well. However, as they are not bound to one single application and they have a broader knowledge of mathematics, they will be able to teach more successfully even theorems not presented in every detail. Therefore, at the Technical University Budapest we are teaching in mathematics a lot of branches of our science, the very basic ones in all details, the other ones on a large scale but trying to show the logical relations among them and mentioning the most important fields of applications.

**Helge Elbrønd Jensen, Denmark:** The major problem in teaching at the universities in Denmark is caused by the fact that university studies have gradually changed from an education for the elite, towards mass education. Furthermore, in science and engineering the number of interested students has decreased considerably over the last 5–10 years. Altogether, the *average* student entering university studies requiring mathematics has less knowledge, maturity and academic skills than is optimal for such studies.

The first reaction from the mathematicians is typically to ask for more teaching hours. At a technical university this is in general very difficult to obtain, because there are so many (new) disciplines the students should learn. Another natural reaction is to ask for higher entrance requirements to university. But this may also be difficult because both the society and the university itself wants a certain number of engineers to graduate each year.

At my university we have decided to face facts and teach the compulsory courses in a way which is suitable for the actual average student, i.e. by reducing the curriculum (cutting away some sub-disciplines) and abandoning the detailed axiomatic, proof-oriented approach to mathematics. Instead we focus on basic concepts and results in calculus and linear algebra, and try to teach the students the meaning and content of the concepts and their significance in solving a variety
of problems. We also offer a number of courses in mathematics which are more or less voluntary for the students. In these courses we teach in a different way and put emphasis on precision, proofs, axiomatics and the abstract approach. In this way the students can learn mathematics from different points of view and in my opinion this works pretty well.

The mathematical needs differ for various types of engineering students, and it is an advantage both for the teaching and the mathematics that students following a specific course have a serious wish to learn the subject of that course. Particularly, I would like to emphasize that we have many very good students, and for those students it is possible to reach a level comparable to a normal university degree in mathematics.

The second great problem in the teaching of mathematics is to find out how to incorporate the computer. The possibilities of the computer make it less important for an engineer to master detailed calculations of the traditional kind, and consequently we need not focus so much as before on these matters. But then we have a pedagogical problem. The working of many exercises, calculations and examples is a decisive part of learning abstract ideas, and therefore, if we pay less attention to doing things with paper and pencil (and the head!), there is a great risk that more general understanding will subsequently decrease. The right balance must be found.

What really matters in the applications of mathematics at university level is not the ability to carry out large calculations by hand, because the computer can do this, and it is not the ability to carry out complicated long proofs, because this often, for the engineering students, adds very little to understanding. What really matters, in my opinion, is the ability to understand the mathematical concepts and the meaning of the mathematical results, and the ability to carry out mathematical thinking at an abstract level. The computer should be used as a tool to train this ability and not as a tool to avoid it.

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New challenges for the teaching of mathematics

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In the not too distant future, if present trends continue, a large proportion of the population of many countries will go to university, which will become the common place for a tertiary education. Moreover, many adults with university studies behind them will return to university, from time to time, to be retrained in work connected with their profession. In such countries, the university will become a place for continuing education.

In these circumstances the distinction between mathematics for all and mathematics for those who will go to university might somehow lose its meaning. But a distinction still has to be made between those who will attend science and technology faculties and those who will attend all other faculties. When selecting the
mathematical content in the curriculum at the secondary school level, it will therefore become increasingly important to choose those units of mathematics which foster the right skills, abilities and attitudes for a meaningful and useful tertiary education.

In Europe most countries have a high proportion of the population attending secondary school. In Denmark, for example, about 60% of teenagers currently complete a secondary school education. In countries where education for the majority of the population currently ends in early grades, it is to be hoped that, in the near future, it will be possible to extend mass education, at least until middle schools. In such a mass oriented education, I think one should avoid too much abstraction and too much formalism; and teaching methods and curricula should be linked to local culture and tradition, and not just adopt curricula from some leading country.

One big problem will probably continue to be a point of discussion, namely from which grade in school should there be different curricula reflecting the future choices of the students? Probably it should take place somewhere between grade 8 and grade 10 in school. For those students who have more interest in mathematics, the conceptual and structural aspects of mathematics should be stressed. For those who are less mathematically motivated, it is advisable to put more stress on the applications of mathematics.

In the 21st century computers will be part of daily life and teaching at all levels might to a large extent be based on interactive computer programs. This will probably be beneficial more for those who are already motivated and successful than for the students who have difficulties with the traditional ways of teaching. It will be an important task to rethink the role of teachers and the importance of social contacts in such an educational environment. In these didactical considerations one should not exclude the possibility that the changes might not go very much further than making available a very powerful teaching aid, like a ‘dynamic overhead’. In any event, the understanding of concepts and the ability to do creative thinking with these concepts using all available tools will be increasingly important. Since computer programs make strong use of visualization and visual thinking must be learned, I predict that interest in basic geometry will grow.

Comments

Gerhard König, Germany: One of the most important tasks in mathematics education today is the revision of curricula and teaching strategies to take advantage of the expanding electronic technology currently available to both teachers and students.

There are many reasons for those involved in mathematics education to reconsider mathematics teaching and its perspectives for the 21st century:

(a) The challenges of new technologies and their use in teaching;
(b) Developments and trends in the mathematical sciences;
(c) The penetration of informatics/computer science into the curriculum;
(d) Mathematics as a technology and its contribution to general education;
(e) New results in the theory of learning.
The most important of the above reasons is the impact of new technologies (computer algebra systems, supercalculators such as the TI-92 or other graphics calculators), which will undoubtedly result in curricular changes in the near future. The prevailing opinion is that not only a change in curriculum, but also a reform of teaching methods, is necessary. Others argue that with the help of new technology and the use of better teaching methods the current curriculum is quite satisfactory. The question of whether curriculum or pedagogy is the major issue of education remains debatable.

The new technologies open a window on creative learning, and computer algebra systems and graphic calculators may help students to achieve a higher level of learning. But many questions remain open. When mathematics courses are built around the use of computers they tend in general to be more demanding. The downgrading of routine skills will make mathematics still more difficult for everyone in the future. Is this actually desirable? Can more interesting mathematics produce more motivation as compensation? Furthermore, curricular and pedagogical changes cannot occur without accompanying changes in student assessment. Turning to the teachers, the continual development of new technologies requires teachers to constantly enhance their technological skills. How many of them will do this? Where do teachers find the time to learn to use these new tools and the time to train students to use them (if necessary)?

And finally, the problem on the role of basic skills is quite open. Which basic skills are necessary in the future is a matter of discussion in several countries. In this connection it is important to note that it takes time to develop and evaluate major curricular reforms. I think it will take a lot of experiment, research, discussion, and conferences before we have a clear picture of how we can integrate the new tools in the teaching and learning processes.

David Salinger, UK: What follows is a personal view of the situation in England and Wales. (Scotland has a separate educational system.)

With the expansion of knowledge in the twentieth century has come a growth in uncertainty as to what constitutes a cultural core. Mathematics has to jostle in the school curriculum with a host of other subjects. Even the notion of what constitutes mathematics has changed. So mathematics occupies a smaller slot in the school timetable and the subjects ‘arithmetic, algebra, and geometry’ are diluted to allow place for ‘data processing, statistics, project-work’. Geometry has been de-natured to such an extent that proof is effectively absent from it. This is not an accident: teachers found that only the best students gained anything from the study of rigorous Euclidean geometry and campaigned successfully for its removal.

As a result, university mathematicians are querying whether the subject called “mathematics” is, any longer, mathematics. The absence of proof and, hence, of coherent development of topics, leads pupils to view the subject as an unstructured collection of useful results.

However, “mathematics” seems to have become a more attractive subject at school and, in recent years, more students have opted to continue “mathematics” beyond the age of 16. It would not serve a useful purpose if a reform of the school curriculum, inspired by mathematicians’ concerns, discouraged students from pursuing mathematics beyond the school level.
Meanwhile, university mathematicians, and other users of mathematics, have been active in several ways. They have, at last, sought to have an official voice in the processes which determine the school curriculum. They have sought to engage in a public dialogue with teachers of mathematics and the theorists of mathematics education. They have sought to stimulate interest in mathematics through masterclasses and competitions. And they have changed university first degree course to give students more time on the early stages, while preserving the level of degree by allowing the better students to take four, rather than three, years to complete it.

Though much remains to be done, I am optimistic about the outcome.

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Feedback from mathematics education to mathematical research

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It is all too obvious that mathematical research exerts a strong influence on mathematics education. It is maybe less obvious that conversely there exists an equally strong feedback from mathematics education to mathematical research. In this short contribution, I will try to illustrate at least one of the many aspects of this feedback: the effect of mathematics teaching at school on the enrollment of students at university in mathematics classes, and on the attitudes of mathematics researchers towards the most innovative parts of our discipline.

At the end of secondary school, pupils who wish to continue their studies at college or university must choose a faculty and a curriculum. This choice is often determined only by a superficial knowledge and by a purely emotional perception of the characteristic features of the various disciplines learnt at school, as conveyed to them by their teachers from early grades.

If at primary and secondary levels pupils have experienced mathematics as a mere technical tool for utilitarian needs, or as a formal, unchanging and old-fashioned discipline, it is likely that even those pupils who might be potentially interested in studying ‘true’ mathematics will prefer to embrace different fields of study, in the expectation of subsequently taking up a job or a profession which seems in their eyes more challenging than that of a mathematician, e.g. engineer, physicist or computer scientist. Unfortunately we must admit that we mathematicians are less able than engineers, physicists and computer scientists in conveying to the larger public a correct perception of what a mathematicians’ work really consists of!

Conversely, if mathematics has been taught to pupils from the beginning as a living subject, highlighting its ongoing dynamical evolution and its capability of contributing to the solution of a huge variety of challenging problems, both practical and theoretical, it is more likely that at least some of the brighter pupils will be attracted to enter into further mathematical studies.

But what about those (rather few) students who have already chosen to become mathematicians? Whatever their initial motivation for this decision may
have been, at some point in their university career they must make further choices concerning their future specific mathematical research fields. These choices, as well as their personal research style are going to be heavily influenced, whether they are aware of it or not, by the “imprinting” they have received during their former studies. If mathematics has been shown to them (as is common even at university!) mainly as a polished end product, where axioms, definitions, theorems and formulae follow each other in a neat order, it is no wonder that they will in turn be most attracted to research in well-established mathematical fields, whose structures have already been wholly shaped and codified. And they will be reluctant to apply themselves boldly and inventively to topics still in gestation, where large parts of a global theoretical framework have yet to be formalized, in spite of the fact that in new fields many more challenging problems remain to be investigated. And if it happens that these same researchers will teach in their turn future primary and secondary teachers, they will be inclined to perpetuate a formal and static conception of mathematics.

At all levels, including universities, a more ‘problem oriented’ approach to the teaching, learning and assessing process in mathematics could be more suitable to give to new researchers more confidence in their creative attitudes and to encourage them to enter into paths not yet completely marked out. Thus the importance of making guesses and conjectures before passing to formal proofs would be stressed, and even errors would not be seen as an absolute evil, but rather as an unavoidable component of progress in knowledge.

Is it not true that the value of a new discovery is due to its unpredictability and to its capacity to link together aspects that seemed until then unrelated, more than to the mere subtlety of the techniques involved?

Of course formalization and rigour are important, but they are not the only goals of doing mathematics. Formalization without meaning is worthless. Rigour is a goal to which to aspire, but absolute rigour is not achievable and moreover, even if it were, the result would be endlessly boring, just like a computer program where too many technical details end up in hiding, rather than clarifying, the main underlying ideas.


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The contribution of the didactics of mathematics to our understanding of mathematics as a science and a subject

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The teaching and the development of mathematics

In the history of mathematics, it is widely recognised that major conceptual developments in mathematics as a science have been stimulated, and in several cases even generated, by needs arising in the teaching of mathematics. The fact that mathematics has had to be taught, at all levels, to interested, enthusiastic, critical, demanding, and reflective students has required teachers, lecturers, and textbook authors to structure, organise and clarify their thinking and exposition of mathematics in order for their students to grasp, understand and learn (acquire) what is taught. This, in turn, further leads to the necessity, in the teaching of mathematics, to lay a foundation for the exposition through the definition of concepts and objects, the investigation of their properties, logical relationships and hierarchies, the establishment of methods and criteria of justification, as well as the determination of terminology, notation, and symbols.

Thus, the teaching of mathematics at various levels provides instrumental contributions to the conceptual formation and development of mathematics. This was the case, e.g., in ancient Mesopotamia (demonstrated in the work of Jens Høyrup), in classic Greece (Euclid’s Elements was probably written primarily as an advanced textbook), in the 19th century in Cauchy’s and Weierstrass’ university lectures, and in the first several stages of the Bourbaki project of our century in relation to mathematics taught at the university level. Also, throughout the 20th century, the production of authoritative, advanced textbooks introducing new topics or theories for the first time has not only served the purpose of making new developments accessible to a wide readership of newcomers to the area, but quite as much of laying the conceptual, logical and terminological foundation of the new development in a way convincing to the mathematical research community. In a way, such textbooks, rather than the initial underlying research papers, can be perceived as ‘certificates of citizenship’ of innovations in mathematics as a science.

Of course, these deliberations are not meant to imply that all major developments in mathematics are driven by the needs of teaching. That is simply not the case. What is the case, however, is that during the last hundred years, at least, the emergence of any major new development in mathematics has been accompanied by the publication of such almost ‘canonical’, first comprehensive introductions to the field, normally written by the ‘field creators’ themselves. In other words, mathematical exposition, by serving the dual purpose of reclaiming new mathematical land to the discipline and of introducing mathematical topics or theories to new recipients (whether students or colleagues), provides an essential and intimate link between the production and teaching of new mathematical knowledge.

Mathematics elucidated by the didactics of mathematics

So far, we have been dealing with the teaching of mathematics only, not with the didactics of mathematics as the scholarly and scientific field of research and development that studies the teaching and learning of mathematics. The fact that
the teaching of mathematics has (had) an important impact on the development and constitution of mathematics as a science does not automatically imply that the same is true with the didactics of mathematics. In what follows I shall, by giving three general examples, argue that the didactics of mathematics does, in fact, contribute to our understanding of mathematics as a science and a subject. This is not equivalent to saying that this field generates innovations in mathematics as a science. However, to the extent that the didactics of mathematics has valuable contributions to offer to our understanding of mathematics, it would not be unreasonable to expect it to give rise to an impact on the mathematics community and its professional practice.

The first example is closely related to the way in which mathematical teaching and exposition has influenced the conceptual constitution of mathematics. In the first stage of the didactics of mathematics, the main focus was on the study and construction of curricula. The investigation of conceptual networks and hierarchies, from both logico-systematic and cognitive perspectives, used to be a predominant activity – in German-speaking countries usually named “Stoff-Didaktik”. It was often meant to serve as a foundation for the writing of textbooks and for the presentation of subject matter. Quite an amount of the work done in this area offers penetrating insights into the conceptual foundation and architecture of mathematical topics and sub-disciplines, such as arithmetic and algebraic operations, geometry, and functions, to name just a few. There still is a solid didactical interest in the study of curricula, but as numerous new areas have attracted researchers’ attention in the last couple of decades, curriculum studies no longer occupy a predominant position in the field.

As an offspring of the endeavour to analyse or construct topics of mathematics curricula, a substantial interest in mathematical cognition has emerged and developed since the 1970s. Rather than focusing on the logico-systematic aspects of larger networks of concepts, studies in cognition investigate the ways in which students perceive, understand and acquire (or the opposite!) specific mathematical notions or concepts, or clusters of interrelated notions/concepts, e.g. the notions of limit, continuity or differentiability, the concept of fraction, the notion of proof. Several such studies are stimulated by the empirical observation that students’ errors and misconceptions more often than not display characteristic patterns within and across individual students, patterns which may provide access to the ways students actually (may) think about the mathematics at issue. On closer analysis, there is, not infrequently, found to be a substantial amount of rationality in students’ notions and concepts, even if the rationality happens to rely or act upon insufficient or wrong conceptions that call for remedy. Such studies not only help us understand how students think with respect to mathematics and how they acquire mathematical knowledge. They also serve to uncover, elucidate and clarify the nature and characteristics of mathematical notions, concepts, and thinking per se, and thus to enrich our understanding of mathematics as a science and a subject. Intimately linked to this is the understanding of the nature of mathematical problem solving, of proofs and proving, features of the mathematical discourse (language and symbolism) and so forth, all of which are intensive areas of study in the didactics of mathematics.
While my first two examples address matters that are, primarily, interior to the edifice of mathematics, my third and final example involves exterior matters as well. As mathematics education is being given to human beings who live and (are to) work in society, the didactics of mathematics has to address matters pertinent to the position, role, and function of mathematics in the world (i.e. nature, society, and culture). While it is, of course, impossible to come to grips with mathematics in its relations to the world without considering mathematics as a science, it is equally necessary to consider mathematics from outside. In other words, meta-studies of mathematics are also required. The position of mathematics in the world is determined by its five-fold nature as a pure science, an applied science, a system of instruments for social practice, a field of aesthetics, and last but not least, a subject of teaching and learning. The latter aspect is a reflection of the four former aspects, of which the first three are the most important ones. It is an essential task for the didactics of mathematics to conduct scholarly and scientific investigations into the position, role, and functions of mathematics in the world (which is not to say that the didactics of mathematics should be alone in assuming responsibility for this task). Issues such as ‘where is mathematics located in the practice of society?’, ‘where and how is mathematics applied in other subject and practice areas?’, ‘what categories of mathematical competence are needed in society, and who should possess which competences?’, ‘what are the effects of mathematical applications and modelling for the organisation and government of society?’ , ‘what sorts of public appreciation of and insight into mathematics are desirable, and how can they be achieved?’ are all crucial in the didactics of mathematics. Answers to them are not only relevant to mathematics education but also to our understanding of mathematics as a science and a subject. The significance of such understanding deserves to be appreciated by the mathematical community at large.

Mathematics education towards mathematics

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The number of human beings involved in mathematical research is of the order of $10^5$; in mathematics teaching, $10^7$; and in mathematics learning, $10^9$. In no other science is the relative importance of education so large. Why is this so? What should be taught, learned, and how? These basic questions for mathematics educators and didacticians are addressed to research mathematicians as well. My personal experience is that educational studies, such as the series of studies performed by ICMI since 1985, are very stimulating in order to investigate and grasp some essential features of mathematics as a science. Before giving examples, let me say that educational concern is part of the mathematical tradition, from Euclid to Bourbaki's *Eléments de mathématique* through Euler’s *Lettres à une princesse allemande*, Laplace’s *Leçons à l'Ecole Normale*, Cauchy’s *Cours de l'Ecole Polytechnique* and Felix Klein’s *Elementarmathematik vom höheren Standpunkte aus*. To a large extent, the history of mathematical education is a very
good introduction to the history of mathematics, not only because some of the best mathematicians of all times and countries contributed to mathematical education, but also, and mainly, because social effects of and impulses to mathematics can be deciphered from school programs and university courses more easily than from research articles.

Let me come back to the present time. Among the first ICMI studies I would consider *The influence of computers and informatics on mathematics and its teaching*, *Mathematics as a service subject*, and *The popularization of mathematics*. The three of them dealt with key issues in or about mathematics education. At the same time they provided an interesting new view of mathematics itself.

The idea that computers and informatics should have an effect on mathematics was not new in 1985; I heard prophetic statements thereabouts by J.S. Haldane when I was a student, in the late 1940s. However, the first international study on this topic – including the effect on concepts, trends of research, style, methods and proofs – was the ICMI study. It is still a good reference now.

Mathematics as a service subject does not mean second rank mathematics, but mathematics taught for a specific purpose. The choice of subject matter and the method of teaching depend on the main interests and future needs of the students. This forces us to consider mathematics not as a tree rooted somewhere in set theory, but as a graph with many entrance doors. Moreover, discussion showed that almost every mathematical notion can be introduced and developed at different levels with different meanings. Some oppositions between mathematics and physics may prove irrelevant; for example, a training in geometry is needed by physicists because of the deep and intimate link between physics and geometry.

The study on the popularization of mathematics was an opportunity to collect information and to suggest future action – actually the situation is better now than some years ago, and it has been claimed that popularization of mathematics has become popular. As a by-product we pointed out a specific feature of mathematics as a science. For there is one more reason to popularize mathematics more than any other science: not being related to any specific domain of the natural or social world, mathematics comes from everywhere and may go in unpredictable directions, as soon as people are aware of its power. If mathematics is not widespread enough, part of its creative power is lost.

These are very general examples. It would be possible to start from any theme of the ICMI studies, mathematics teaching in the 1990s, mathematics and cognition, assessment in mathematics, gender in mathematics education, research in mathematics education, and show how it involves some fundamental aspects of mathematics as a science.

As a personal conclusion, I should like more mathematicians to be involved in the field of didactics. I have stressed the possible benefit for mathematicians. I am sure that it could help didacticians as well.