Solid Findings in Mathematics Education: Living with Beliefs and Orientations – Underestimated, Nevertheless Omnipresent, Factors for Mathematics Teaching and Learning

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Triggers

Already in the 1940s researchers were investigating the question of whether attitudes may have some influence on the *reception of mathematics* and *its assessment*. Articles on this subject provide an affirmative answer, which may be astonishing for an expert. In addition to the rather obvious affective side, it was learnt in the 1950s that, even more, *subjective understandings* of mathematics and mathematics teaching are influencing the reception of the content and may lead to an unsatisfactory assessment of mathematical topics. Even more, they may affect engagement structures (see Goldin et al., 2011).

In 1983, Alan Schoenfeld published an article dealing with important driving forces beyond the purely cognitive. His insights contributed to a growing research field, whose phenomena were not well understood but accepted as observed and highly influential. Schoenfeld pointed out two years later: *These phenomena were, on the whole, negative; for example students were turned off mathematics by the drill-and-practice during the heyday of the back to basics movement.*

For a short while, it seemed that propagating problem solving as a curricula guideline would be a promising alternative. However, curricula favouring problem solving, first established in the USA in the 1980s, soon turned out to be a failure. What were the reasons?

It was Marta Frank who studied in her PhD (1985) the (potential) incompatibility between problem solving and teachers' and students' inherent *beliefs*.

What are beliefs? Referring to a definition of Schoenfeld: *Beliefs are mental constructs representing the codification of people's experiences and understandings.* Beliefs have already been studied by educational psychologists. It was Green (1971) who emphasised that by communicating knowledge, to shape behaviour, e.g. doing mathematics, the instruction and teaching processes are always accompanied by generating and shaping beliefs. Beliefs are not isolated, arbitrary subjective constructs; they are part of larger systems. To quote Green (1971): *Nobody holds a belief in total independence of all other beliefs* (p. 41). Beliefs always occur in sets or groups. Beliefs are clustering. So the terminology used by Schoenfeld and offering the term *mathematical worldviews* is figurative.

Thus, we have to accept mental subjective structures beyond the purely cognitive, which are widely unknown and not easily influenced. To be precise, Frank and many others stated that students may not be able to become better problem solvers unless they change their beliefs about mathematics and build up adequate worldviews on mathematics.

As Green (1971) pointed out, beliefs are not inherent only in mathematics education. Beliefs occur everywhere; we are living by beliefs, often not being aware of beliefs that may occur e.g. as prejudices. In particular, as shown below, in research mathematics much is also based on beliefs.

Folklore beliefs

We list some folklore beliefs, which can be encountered in nearly all mathematics classrooms across the world and which are still alive despite many teachers gradually becoming aware of these misleading views and fighting against them.

- "Mathematics is created only by very prodigious and creative people; other people just try to learn what is handed down."
- "One can not engage in doing mathematics by oneself. One requires instruction."
- "Mathematics is unjust, because some students are privileged, while most are at a disadvantage."
- "Dependence on the teacher is important in mathematics. However, if the teacher cannot 'clarify', there is little chance that the student will understand mathematics."
- "Deduction is of the highest importance in mathematics. Thus structures and proofs play a major role in mathematics."
- "There is, in this sense, nothing worse than making mistakes in mathematics. Mistakes are damned."
- "A mathematical task always has only one solution, so I hate mathematics, because there are thousands of wrong answers, however only one correct answer."

The last quotation from a student's interview (age 10) makes clear that inadequate or insufficient worldviews on mathematics influence our attitude towards mathematics directly.

- "Any mathematical problem can be solved in two minutes, or not at all."

Again this restricted understanding on mathematical problems partly explains the failure of the innovative approach of problem solving. Again we have to ensure in advance that the beliefs about mathematics as a whole are not misleading. Research has proven that these beliefs of that type, sometimes called estimations, assumptions, prejudices and half-known knowledge, drive the thinking of students in many classrooms around the world.

But not only the behaviour of students leaves something to be desired; the same laws apply to teachers and again many folklore beliefs can be found in various papers. We refer to an article of Tobin and LaMaster (1992) which states: *However, what became apparent was that teachers implemented the curriculum in accordance with their own knowledge and beliefs and did not necessarily do what curriculum designers envisioned. Several studies ... indicated that teachers do what they do in classrooms because of their beliefs about what should be done and how students learn.* (p. 115)

Beliefs in research mathematics?

At first glance, the reader might accept our statements but be reluctant to accept a similar situation also in the field of mathematics. To be honest, our research is often driven by other research and there is nothing sinister in this. But some of the beliefs on mathematics are rather global, among them philosophies. Platonism, for example, is a belief, a worldview or a philosophy; the same is true about formalism. Hersh (1997) describes devastating influences of certain philosophies. In an introductory paper of P. J. Davis (1972), he lists the various, often non-reflected implications of a 'platonic mathematics – philosophy'. Mathematicians are sometimes disputing different underlying philosophies; however other philosophies (see Hersh, 1997) imply different grounded assertions and worldviews.

It should not be ignored that philosophies about mathematics are also influencing our pedagogy, an effect René Thom (1973) phrased in the following way: *In fact, whether one wishes it or not, all mathematical pedagogy even if scarcely coherent, rests on a philosophy of mathematics* (p. 204). Although Thom's statement has to be acknowledged as a hypothesis from an educational perspective, there is much evidence for the linkage between mathematics and mathematics education.

To use the terminology of belief research, philosophies about mathematics can be understood as beliefs about the (large) 'belief object' mathematics. But there are also beliefs about smaller objects; take, for example, the definition of continuity.

We all know that these beliefs, the associations or the students' 'concept image' are often not adequate for explaining some difficulties in our university classrooms. But the list of examples is endless: think of vector spaces, differential equations or skew fields, just to mention three mathematical objects or terms. Think of the formula $z^n = r^n(\cos n\varphi + i \sin n\varphi)$ with z a complex number; discuss it with your students and you will realise that there are beliefs associated and, very often, emotions or affections linked to them. Finally, use a 'smaller' belief object, namely the equation $(-1)(-1) = 1$ and again you will encounter beliefs. Feel encouraged to start a discussion on this equation with students at school. You will experience what a researcher once described with the metaphor: Investigating beliefs is like picking out a single spaghetti thread and soon you will realise that more threads are glued to it.

Research on beliefs and orientations

What should be learnt? Where do we stand today? What is worth being mentioned?

Beliefs are *no longer a hidden variable* acting influentially as they were described at the beginning of the century (Leder, Pehkonen, Törner, 2002). Most mathematics education researchers are aware of beliefs. While in 1992 a first survey article by Thompson about the influence of mathematics teachers' beliefs was published, today every handbook contains a chapter dealing with beliefs – and each international conference has its beliefs session.

Nevertheless, 'beliefs' are a fuzzy construct and it seems unsatisfactory to mathematicians that there is generally no internationally accepted definition. Thus, when reading a paper dealing with beliefs, you often have to question what the author's (personal) definition on beliefs is. However, Pajares (1992) stated that the most fruitful concepts are those to which it is impossible to attach a well-defined meaning (see an analysis in Törner, 2002). Thus, different authors are often using slightly different 'definitions'.

Meanwhile, Schoenfeld (2010) has introduced a new term, namely *orientations*, which is a much wider construct and which is also integrating individual goals that are closely connected with one's beliefs. Beliefs and goals often explain themselves mutually.

There is still a dispute in the community of whether we should include affections and emotions within the definition of beliefs. Both alternatives have their 'pros': nearly all beliefs are linked to observable emotions and affections in different degrees; on the other side, the theories of emotions and affections, partly including attitudes, follow different patterns from those on beliefs.

Recently, Goldin et al. (2011) pointed out that so-called engagement structures – for an initial understanding refer to the standard meaning of 'engagement' – can be traced to beliefs and worldviews. However, it is still an unanswered question in educational psychology and sociology whether actions are induced by beliefs and connotations should be associated. Believing and acting accordingly is not a general rule. But the opposite is also not true; very often, actions can be explained through the beliefs of an individual.

What are the implications of belief theory for the teaching and learning of mathematics?

Although mathematics is respected as a theory which does not accept any tolerance in terminology and conceptions, *we are living by beliefs*. We need a finite set of beliefs since they restrict our world to a finite number of patterns. In some cases beliefs are productive and they open our thinking but often they hinder our thoughts and serve as inertia obstacles. Thus we should be aware and sensitive of beliefs beyond the purely cognitive landscapes in mathematics. Nevertheless, we can often rationally explain students' behaviour in a mathematical context because of their underlying beliefs. They are also responsible for our orientations, which are closely linked with goals (see Schoenfeld, 2010): If you are believing something, you will strive for it and declare it as your goal in action and vice versa.

To change beliefs is a hard job and there is no royal road to being successful. How far beliefs determine our behaviour and connotations is still an open question in social psychology. Nevertheless, there are indications that inadequate beliefs may lower the assessment of students while doing mathematics. To prove this empirically is a really hard job since we have to distinguish between *proclaimed* and *possessed beliefs*. However, there is a lot of smoke so there must be some fire somewhere.

Thus, generating and influencing adequate beliefs (which can be extended – if it is necessary – at a later time) is decisive. In mathematics, we do not only have to promote knowledge but also handle and develop beliefs, hopefully to induce adequate beliefs which are productive.

Beliefs – why do we estimate them as solid findings?

In our first article on solid findings in 2011 (EMS Newsletter Issue 81) we listed some criteria for using the quality label of 'solid findings' for the construct under discussion. There is a large body of evidence that all criteria are fitting. Beliefs are not a result of a single research initiative; thousands of articles have proved the 'existence' of beliefs and their influence on teaching and learning of mathematics. Thus, the trustworthiness has to be accepted. Beliefs are an excellent construct in mathematics education to show how universally this idea can be applied. Although beliefs may be dependent on the cultural framework or setting and they may differ in different cultures, they can be verified. Beliefs have also deepened our understanding since they serve as an additional variable, which was hidden before. And as it is happening in many contexts, the more we know about beliefs, the more questions will be asked, e.g. the highly complex structures of how beliefs are influencing our behaviour and our decisions.

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