It is Necessary that Teachers are Mathematically Proficient, but is it Sufficient? Solid Findings in Mathematics Education on Teacher Knowledge

Imagine a middle school mathematics class with 28 students solving algebra problems. Whereas student Andrea, although getting a lot of individual support, often comes up with $(a + b)^2 = a^2 + b^2$, Bettie is close to developing the quadratic formula identity by transforming $x^2 + px + q = 0$ into $(x + p/2)^2 – (p/2)^2 + q = 0$; Charly when dealing with $(a + b)^3$ develops an interest in Pascal's triangle whereas Diane has problems accepting that we can count with a's and b's and not just with natural numbers.

The class is taught by Marta, an experienced teacher with a deep background in content knowledge and in pedagogical content knowledge, and in diagnosing students' mathematical thinking in particular. Like many teachers, she deals with a diversity of students' pre-knowledge.

Her young colleague, Melvin, teaches the second mathematics class of the same grade at her school. He is also mathematically proficient but he has severe problems coping with the complexity of teaching. He mainly teaches from the front of the class which fits the needs of only a small number of his students. The other students would need different kinds of learning environments and specific support. Mina, another colleague, did not study mathematics at university but has to teach this subject due to a shortage of appropriately qualified teachers. She is an enthusiastic teacher but sometimes she is not able to support students' thinking because of her own lack of content knowledge; however, she is aware of this weakness and motivates students to self-study and learning with experienced others. Another teacher, Monte, has severe health problems and is close to burning out. His motivation has decreased over the years and he is frustrated about teaching in general and mathematics at this school in particular.

The school has no tradition of exchanging experiences among subject teachers. Thus the ideas, strengths, problems and strategies of Marta, Melvin, Mina, Monte and others remain individual. The principal does not really value mathematics; thus his interest in and support for mathematics teachers is limited. The mathematics colleagues never meet to share lesson plans or to discuss their understandings about what good mathematics teaching is and how the school might improve students' interest and competence in mathematics.

Nearly every teacher at this school goes their own way, with individual variations of knowledge and with idiosyncratic interpretations of the national curriculum. The teachers themselves do not seek collaboration and few of them have experienced the benefits of sharing educational opportunities and challenges in their own teacher education. Even if a new ambitious and mathematically competent teacher starts at this school, they may not be authoritative enough to change the "culture" of isolated teaching and the poor "general conditions" for mathematics teaching at this school.

The situation sketched above can be found in thousands of variations all over the world, influenced by specific cultural, geographical, historical, socio-economic and political circumstances. The story illustrates that content knowledge is at best a necessary, but far from being a sufficient, condition for "good teaching" (see, for example, Wilson, Cooney & Stinson, 2005). Teachers need more than mathematical knowledge; they also need to communicate with and learn from each other and to get adequate internal and external support for their task. Thus mathematics teaching, in addition to mathematical considerations, also needs to take into account a variety of individual, social and organisational aspects. In other words: *content* is important but so is *community* and *context*.

When reduced to a purely mathematical point of view, mathematics teaching seems to be well defined. However, when taking into account the individual, social and organisational aspects of schooling, the picture becomes more complex: not only do individual students' and teachers' knowledge and interests vary greatly but also the forms of professional communication, of teacher education and of the context of teachers' work at schools.

Research in mathematics education, in particular in mathematics teacher education, underlines the fact that *content, community* and *context* are decisive factors of mathematics teaching and teachers' learning (see, for example, Krainer, 2011). This is mirrored in a variety of research findings, from which some exemplary findings are sketched in the following.

Content: Mathematical knowledge for teaching

Knowing mathematics is naturally an essential prerequisite for teaching mathematics. However, efficient mathematics teaching, whoever the students are, also requires other kinds of knowledge and skills (Ball, Hill & Bass, 2005) and thus appropriate learning opportunities. This insight is reflected by the introduction of different types of teachers' knowledge in mathematics education research. The most prominent typology describing mathematics teachers' knowledge goes back to Shulman (1987), who differentiated six different types, from which Content Knowledge (CK), Pedagogical Content Knowledge (PCK) (meaning the specific knowledge that is needed for teaching mathematics) and Pedagogical Knowledge (PK) are the most important ones. However, teachers' knowledge can also be regarded as knowledge about learning and teaching processes, assessment, evaluation methods and classroom management; other foci are expressed by the notions of attention-based knowledge or knowledge of mathematics for teaching. A recent overview is given in the first volume of the *International*

Handbook of Mathematics Teacher Education (Sullivan & Wood, 2008).

Many studies and surveys – in several domains of teacher education – indicate the importance of Pedagogical Content Knowledge. In a recent German study (COACTIV project – see, for example, Baumert et al., 2010) the researchers show that mathematics teachers from Gymnasium (who traditionally get a strong CK background in teacher education) outscore teachers from other secondary-level school types on CK but also on PCK. Gymnasium teachers also exhibit a higher degree of cognitive connectedness between the two knowledge categories. Also within the COACTIV project, tests of secondary mathematics teachers' CK and PCK have been developed and implemented in a sample of teachers whose classes participated in the PISA 2003/04 longitudinal assessment. The study shows that students taught by teachers with a high PCK showed better PISA results than those of other students. In particular, they showed within one school year a greater increase in achievement. Reasons given were that teachers with a high PCK design their teaching so that the students are more actively cognitively engaged. This was shown, in particular, through the analysis of tasks the teachers used in the classroom. The effect seems to be specific for PCK since a similar correlation between teachers' CK and students' achievement was not found. This was confirmed in further analyses: PCK has "greater predictive power for student progress and is decisive for the quality of instruction" (Baumert et al., 2010, p. 164). Knowing mathematics is important but not enough to teach mathematics effectively.

Good teachers also have insights into the mathematics content their students will learn that allows them to identify misconceptions, set up lessons in which students make connections and provide students with questions and prompts that help them access important concepts. Studies show that supporting student teachers' reflections of their own thinking processes leads them to understand better and improve their instructional practices. Meaningful mathematics-related activities and reflections on them support prospective and practising teachers' growth (see, for example, Even & Ball, 2009). Such activities can include solving mathematical problems themselves, choosing appropriate content for the preparation of a lesson, investigating students' mathematical learning or observing and reflecting on one's own or other colleagues' teaching.

This principle holds in particular in Japanese lesson study (see, for example, Hart, Alston & Murata, 2011), a mode of teacher education whose efficiency has been established in studies in Japan and elsewhere. A typical *lesson study* process contains 4–6 steps, having a *study lesson* as its focus: 1) Collaboratively planning the study lesson; 2) Seeing the study lesson in action; 3) Discussing the study lesson; 4) Revising the lesson (optional); 5) Teaching the new version of the lesson (optional); and 6) Sharing reflections about the new versions of the lesson. The vast majority of elementary schools and many middle schools in Japan conduct lesson study in all subjects. Many schools solicit the support of an external advisor (most often instructional superintendents, sometimes experienced teachers on leave or university staff). Some schools even produce written reports about their work. For example, in the early 1990s the National Institute for Educational Research compiled over 4,000 reports written by teachers every year.

Similar approaches are carried out in other Asian countries, for example within the Keli project in China (see, for example, Huang & Bao, 2006), where exemplary lessons are developed and disseminated at the school and district level. It is assumed that the success of Asian countries in international comparative achievement studies like PISA and TIMSS can partly be explained by this focus on planning, observing, reflecting and revising lessons in a joint effort. This leads to the factor "*community*".

Community: Collaboration among mathematics teachers

Research on "successful" schools shows that such schools are more likely to have teachers who have continual substantive interactions and that inter-staff relations are seen as an important dimension of school quality (Reynolds, Creemers, Stringfield, Teddlie & Schaffer, 2002). The latter study illustrates examples of potentially useful practices, of which the first (illustrated by a US researcher who reflects on observations in other countries) relates to teacher collaboration and community building (p. 281): "Seeing excellent instruction in an Asian context, one can appreciate the lesson, but also understand that the lesson did not arrive magically. It was planned, often in conjunction with an entire grade-level-team (or, for a first-year teacher, with a master teacher) in the teachers' shared office and work area." Such a collaborative habit needs to be fostered from the very beginning of pre-service teacher education. This statement holds for primary and secondary teachers but also for university teachers (see, for example, Nardi, 2008).

One key characteristic of Japanese lesson study is that it is "*collaborative*" (Murata in Hart, Alston & Murata, 2011). She indicates that novice teachers who experience the lesson with experienced teachers are apprenticed into the profession through participation. This community aspect goes beyond the idea of collaboration among individual persons. It is about a way of further developing a profession (by engaging novices into serious academic activity and thus fostering identity building). Community building is not confined to teachers but also extends to students: the vision of Japanese teaching is organising collective thinking, focusing on student presentations and discussions. The succession of single work, group work, plenary discussion in classrooms and teachers' comments helps to balance individual and social learning (Krainer, 2011).

Also in the Chinese Keli project (Huang & Bao, 2006) collaboration plays a decisive role. In this project, a "community" consisting of experts, teachers and researchers is formed and the teachers improve their teaching action and upgrade their professional theory through unfolding the Keli process in cooperation with the members of the community.

Several studies worldwide indicate the positive impact of teachers' collegial learning. For example, Jackson and Bruegmann (2009) use longitudinal elementary school teacher and student data for documenting that students have larger test score gains when their teachers experience improvements in the observable characteristics of their colleagues. They show that teachers' students have larger achievement gains in mathematics and reading when they have more effective colleagues.

The Learning Communities in Mathematics (LCM) research and development project in Norway (see, for example, Jaworski, 2008) brought together teachers and didacticians (the term that the team preferred to use for the teacher educators) to work together as both practitioners and researchers. It involved a team of 14 didacticians working with eight primary and secondary schools. Among others, Jaworski (2008, p. 326) highlights that "teachers suddenly came to see, through their study of students' thinking and activity in algebra, how they could explore in their school environment ways to develop teaching and learning; didacticians saw the nature of a task that could lead to teachers' effective recognition of the nature of school goals for students' development and learning in mathematics".

At a university in Finland, new ways of teaching a lecture course and of giving peer support to beginning mathematics students doubled (at least) the number of students passing certain important courses compared to the earlier prevailing level (Oikkonen, 2009). Several studies indicate the new possibilities opened by digital networking (see, for example, Gueudet, 2010).

The importance of the "community" factor is mirrored by a stronger emphasis on sociological and socio-cultural aspects of teaching and teacher education. Concepts like didactical contract (which will be dealt with in the next EMS Newsletter), inquiry community, institutional constraints, negotiation of meanings and norms, organisational development, sharing of knowledge and systemic learning are used that go beyond cognitive views on learning. Lerman and Tsatsaroni (2004) report that between the time periods 1990–1995 and 1996–2001 the percentage of papers in selected journals and proceedings in mathematics education that draw on sociological and socio-cultural theories increased from 3% to about 10%. Two of the four volumes of the first *International Handbook of Mathematics Teacher Education* (Krainer & Wood, 2008; Jaworski & Wood, 2008) deal intensively with the collaboration among teachers or teacher educators and between teachers and teacher educators. When mathematics teachers (and teacher educators) share experiences, ideas, beliefs, competences, challenges and needs, they not only learn themselves but also learn to support others' learning. The processes include working in small teams, communities of practice and loosely-coupled networks. Mathematics teachers need to build identities specialising in students' mathematical learning through collaborative reflection.

Collaboration among mathematics teachers needs adequate general conditions and support of teachers' work. This leads to the next factor, the "*context*".

Context: General conditions and support of mathematics teachers' work

General conditions that are conducive to successfully supporting mathematics teachers include offering learning opportunities where both the factors of content and community are regarded as essential. In particular concerning the work with practising teachers, the context (resources, structures, commitment, etc.) plays an important role. In addition, the influence of principals and other stakeholders of the educational system is indicated by several studies.

In Japanese lesson study, knowledgeable others from outside the lesson study group are sometimes invited, to present observations or links to research or theories, for example. These experts are paid with the help of small external grants. It is common that teachers are internally supported by principals and get release time by hiring substitute teachers. Lesson study groups are supported by townships, boards of education, the ministry, etc. This financial investment is an expression of societal trust in teachers and of believing in school-based professional development. It makes sense to speak of a lesson study "culture" that is conducive to teachers' work. The importance of the context factor is increasingly stressed in studies worldwide. For example, Adler (2000) – in particular looking at the situation in South Africa – worked out the need for conceptualising resources as a theme for mathematics teacher education (e.g. classrooms and learning tools in poor and rich regions are quite different). Nickerson and Moriarty (2005) describe an urban school initiative aimed at teachers' professional development with the goal of increasing teachers' mathematics content knowledge and helping them improve in practice. The research shows that general social and organisational conditions like mathematics teachers' relationships with the school administration and other teachers or the presence of a teacher leader are relevant for (the further development of) good mathematics teaching at schools. Kazemi (2008) stresses the importance of engaging parents as intellectual and social resources. She indicates studies with interventions that have aimed to work with families around mathematics as a complex problem solving discipline and that led to significant increases in families' feelings of empowerment. Cobb and Smith (2008) report on a district development project as a means of improving mathematics teaching and learning at a large scale. Here, research in mathematics education meets with research in educational policy and leadership where institutional settings of schools and districts come to the fore. Also, when investigating the sustainable effects of a professional development programme, context issues like the principal's understanding of leadership or personnel fluctuation play a decisive role.

What do we learn from that? What can be done?

That mathematics teachers need a high level of Content Knowledge (CK) is necessary but not sufficient. There is clear research evidence that other kinds of knowledge are also important; in particular, Pedagogical Content

Knowledge (PCK) is decisive. Furthermore, teachers need to have high social competencies, in particular for two reasons: in order to support students' learning (as individuals, groups and whole classrooms) and in order to learn from other colleagues and other experts. Mathematics teachers need to understand themselves as learning communities that adopt new research findings, share their experiences and discuss ways of improving with others, etc. It is a decisive feature of a (scientific) community to share new knowledge and experiences; this also holds true for the teaching profession. Otherwise, teachers' efforts to improve remain limited to their own classrooms and the wheel is invented again and again. Mathematics teachers also need to understand that communication and collaboration are essential in order to improve not only the teaching of mathematics of one of their classes but of a whole school, or a whole district or even a nation. This needs them to bring in their ideas, interests, knowledge and visions but also their doubts and open problems. Teachers need to reflect on how to improve mathematics teaching at their school, on how to convince the principal to buy a new mathematics education journal or new mathematical software or on how to improve the curriculum. This means to engage in organisational activities, to reflect on the context of mathematics teachers' work at their school and on how to overcome financial and personal restrictions, etc.

Where is the place that mathematics teachers learn all this?

Of course, in the first phase of teacher education, there needs to be a special emphasis on CK, and also on PCK and PK. However, in order to cope with the collaborative nature of the teaching profession, collaboration should also get a clear focus in teacher education. Teachers teach as they have been taught. Therefore, mathematics teacher educators need to regard themselves and act as role models. Also, at least from time to time, they should provide learning opportunities to reflect on the context under which the student teachers learn (e.g. What is the "didactical contract" between the teacher educator and the class? What have mathematics and physics in common or where do they differ, for example, regarding their contribution to education?). Of course, some of these issues can also (and partially better) be learned when teachers have finished their studies and work in practice, and in collaboration with colleagues in their departments or in other forms of professional development. However, it would mean to miss a chance to focus mainly on CK in the first phase of teacher education. Vice versa, professional development should also deal with mathematical issues from time to time. Since not all mathematical fields can be covered in university studies (in particular in primary education), it is important that teachers learn strategies for learning the mathematics they will need.

Mathematics teacher education is a challenge. Teachers need teacher educators who work with them in the same innovative way teacher educators are expecting teachers to teach. These teacher educators need to collaborate with other colleagues like they expect teachers to do. Teacher educators need to evaluate and improve their courses. They need to gather teachers' interests and pre-knowledge because this increases the likelihood that teachers realise the power of having sufficient information about their learners. The list of needed actions and reflections by teacher educators could be easily extended. The stronger this culture of reflection and evaluation becomes established at universities, the stronger it develops at schools (and at school administrations). It is important to investigate the learning of mathematics students and mathematics teachers; however, we also need to put more emphasis on exploring and understanding our influence on mathematics teachers' learning. This kind of teacher education research is a key to supporting the development of mathematical thinking, both of teachers and students.

This paper started with the insufficient situation of mathematics teaching at Marta's school. How could this situation be turned around according to the findings described in this paper? Marta starts to carry out innovations in her algebra class. The number of students loving mathematics increases; others at least begin to lose fear and to construct meaning in thinking mathematically. Together with her colleagues Melvin, Mina and Monte, Marta shares experiences and ideas continuously. Together, they form a kind of learning community, supported by a mathematics teacher educator at the regional university. In collaboration with mathematicians and mathematics educators, mathematical contests, exhibitions and family clubs are established. Student teachers from the university come to this school and collaborate with experienced teachers. Mathematics begins to play an important role at this school. The principal supports the work of his students and teachers and he makes it visible in the district. Representatives of the mathematics teaching staff are invited to speak about their progress at other schools and districts. Considerable research is done with regard to mathematical teaching and learning in the context of this school district which is fed back to teachers, administrators and educational authorities. This movement also influences teacher learning in other subjects. To some extent, the idea of improving mathematics teaching and learning finally turns into a culture of innovation in all subjects. Isn't that a nice (educational) kind of "generalisation"?

Authorship

Even though certain authors have taken the lead in each article of this series, all publications in the series are published by the Education Committee of the European Mathematical Society. The committee members are Ferdinando Arzarello, Tommy Dreyfus, Ghislaine Gueudet, Celia Hoyles, Konrad Krainer, Mogens Niss, Jarmila Novotná, Juha Oikonnen, Núria Planas, Despina Potari, Alexei Sossinsky, Peter Sullivan, Günter Törner and Lieven Verschaffel.

Additional information

A slightly expanded version of this article with a more complete list of references may be found at http://www.euro-math-soc.eu/comm-education2.html.

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